On The Domatic Number of Bipartite Permutation Graphs

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Abstract

This paper studies the problem of partitioning the vertices of a graph G into disjoint set V_1, …, V_q such that V_1, …, V_a are all dominating sets of G. The objective is to find the largest q. We first propose a 0-1 matrix representation of bipartite permutation graphs. Then, a linear-time algorithm for domatic number problem on bipartite permutation graphs is designed.

Key words : domatic number, domatic partition, dominating set, bipartite permutation graph.

1 Introduction

A graph $G = (V,E)$ is *bipartite* if $V(G)$ is the union of two disjoint independent sets of G. A Graph G is *permutation graph* if there exists a permutation π of $\{1,2,3,\ldots,|V|\}$ such that $(i,j)\in E$ if and only if $(i-j)(\pi^{-1}(i) -\pi^{-1}(j)) < 0$. A set $D \subseteq V$ is a *dominating set* if every vertex in $V-D$ is adjacent to at least one vertex in D. Let $GP = \{P|P$ *is a partition of V and every set is a dominating set of G*$\}$. The *domatic number* $dm(G)$ is defined as $dm(G) = max\{size(P)|P\in GP\}$, where $size(P)$ denotes the number of sets in P. A *domatic partition* is a partition in GP whose size equals $dm(G)$.

Finding domatic partition on general graphs is NP-Hard [6]. For this problem, Peng and Chang gave an linear-time algorithm on strongly chordal graphs. Bertossi [1] proposed an $O(n^{2.5})$ for interval graphs and $O(n \log n)$ for proper interval graphs. Later, [10] and [2] gave $O(m+n)$ algorithms for interval graphs, and [8] gave an $O(n)$ algorithm for interval graphs with sorted intervals. In [5], Bonuccelli showed that this problem remained NP-hard for circular-arc graphs and gave an $O(n^2 \log n)$ time algorithm for proper circular-arc graphs. In [3], Haim and Ron showed that it remained NP-hard for chordal and bipartite graphs.

The rest of this paper is organized as follows. In section 2, we give some properties of bipartite permutation graphs and their 0-1 matrix representation. In section 3, we provide a linear time algorithm for this problem.

2 Bipartite Permutation Graphs

In [9], Peng and Chang gave biclique structure of bipartite permutation graphs. A biclique or complete bipartite graph is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. A bipartite permutation graph G can be partitioned into $G_r = B_1 \oplus_{l_1}^{k_1} B_2 \oplus_{l_2}^{k_2} \ldots \oplus_{l_{r-1}}^{k_{r-1}} B_r$, where $B_1,B_2,\ldots,B_r$ are bicliques, called generating base, $(k_1,l_1),\ldots,(k_{r-1},l_{r-1})$ are the common nodes from X and Y partite between any two bicliques. We make use of this structure and refine it in our algorithm. For the sake of simplicity, proof detail can be referenced in [9].

Lemma 2.1 [9] *G_r is a bipartite permutation graph.*

Theorem 2.1 [9] *G is a bipartite permutation graph if and only if there exists a generating base for G.*

Lemma 2.1 shows that G_r and bipartite permutation graph are equivalent. Theorem 2.1 show that every bipartite permutation graph can be converted into $G_r = B_1 \oplus_{l_1}^{k_1} B_2 \oplus_{l_2}^{k_2} \ldots \oplus_{l_{r-1}}^{k_{r-1}} B_r$. The matrix representation in this paper can be easily seen that is equivalent to G_r.

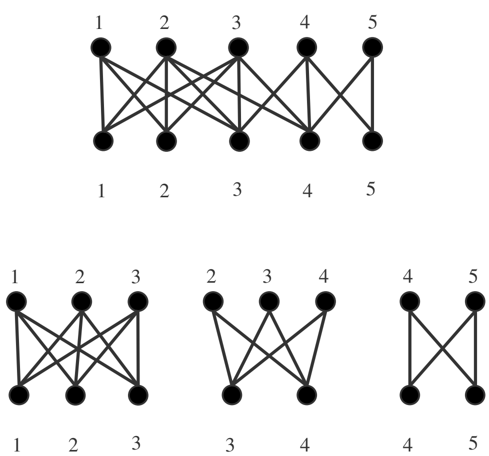

Figure 2.1

Figure 2.1 shows that a bipartite permutation graph can be decomposed into bicliques. It is natural to represent a graph with 0-1 matrix, when we use 0-1 matrix to represent bipartite permutation graphs, it can be easily observed that it has consecutive 1s' property, In Figure 2.2, we show that this two kinds of representation are equivalent.

Y\X	1	2	3	4	5
1	1	1	1		
2	1	1	1		
3	1	1	1	1	
4		1	1	1	1
5				1	1

Figure 2.2

These three rectangles in Figure 2.2 represent three bicliques in Figure 2.1 respectively, and the generating base in Figure 2.1 (In this example, $\oplus_1^2$, $\oplus_1^1$) mapping to Figure 2.2 which is the width and height of the intersection region of any two rectangles.

Now, we give this representation a formal definition, Let $G = (X,Y,E)$, be a bipartite permutation graph, $X = \{1,\ldots,m\}$, $Y = \{1,\ldots,n\}$, construct a m by n 0-1 matrix T where

$$\begin{cases} T_{ij} = 1 & , \text{if } (i,j) \in E, \\ T_{ij} = 0 & , \text{otherwise.} \end{cases}$$

3 Algorithm

Let $G=(X,Y,E)=B_1 \oplus_{l_1}^{k_1} B_2 \oplus_{l_2}^{k_2} \ldots \oplus_{l_{r-1}}^{k_{r-1}} B_r$, be a bipartite permutation graph, define $X(B_i)$ (respectively, $Y(B_i)$) be number of nodes of biclique B_i in X (respectively, Y) partite. Let $P = \{p_0,p_1,\ldots,p_{r-1}\}$ and $Q = \{q_0,q_1,\ldots,q_{r-1}\}$ where

$$p_i = (p_i^x , p_i^y) = (\sum_{j=1}^{i} X(B_j) - k_j + 1,\ \sum_{j=1}^{i} Y(B_j) - l_j + 1)$$

and

$$q_i = (q_i^x , q_i^y) = (\sum_{j=1}^{i} X(B_j) - k_{j-1} + 1 ,\ \sum_{j=1}^{i} Y(B_j) - l_{j-1} + 1),$$

for $1 \le i \le r$.

define $p_0 = (0,0)$ and $q_0 = (0,0)$.

The set P and Q indicate left-top and bottom-right of these intersection regions of two rectangles. In Figure 2.2, the set $P = \{(2,3),(4,4)\}$, $Q = \{(4,4),(5,5)\}$.

For set $P = \{p_0,p_1,\ldots,p_{r-1}\}$, $Q = \{q_0,q_1,\ldots,q_{r-1}\}$, define $region(p_i)$ be the induced subgraph of G, where $V(region(p_i)) = \{ p_i^x , p_i^x +1,\ldots,m\} \cup \{ p_i^y , p_i^y +1,\ldots,n\}$, define $ds(i)$ be the set of nodes label i, $1 \le i \le dm(G)$, define *#stages* be the size of generating base.

Algorithm DP

Input : a bipartite permutation graph $G = B_1 \oplus_{l_1}^{k_1} B_2 \oplus_{l_2}^{k_2} \ldots \oplus_{l_{r-1}}^{k_{r-1}} B_r$.

Output : a domatic partition of G.

BEGIN

FOR $i := 0$
  label (p_{r-1}^x, p_{r-1}^y) to $i + 1$;
  i ++ ;
UNTIL ($p_{r-1}^x + i > m$ or $p_{r-1}^y + i > n$)
IF (the remainder nodes of B_r dominate region(g_{r-1}))
  label these nodes to $i + 1$;
  i ++ ;
END IF
FOR $j := r - 2$ **DOWNTO** 0
  FOR $k := 1$ **TO** i
    IF ($ds(k)$ cannot dominate $region(p_j)$)
      pick two vertices u,v where u,v come from first unlabeled vertex of $region(p_j)$ in X and Y partite respectively, if no such u,v vertices exits, set $u = 0$ or $v = 0$.
      IF ($ds(k) \cup \{u\}$ dominate $region(p_j)$)
        label u to k;
      END IF
      ELSE IF ($ds(k) \cup \{v\}$ dominate $region(p_j)$)
        label v to k;
      END IF
      ELSE IF ($ds(k) \cup \{u,v\}$ dominate $region(p_j)$)
        label u,v to k;
      END IF
      ELSE
        unlabeled $ds(k)$;
        i -- ;
      END ELSE
    ENDIF

merge unlabeled nodes to any dominate set;
END

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4 Correctness of Algorithm DP

We can prove this by induction on *#stages*. Basis step *#stages* = 1, which is the following case :

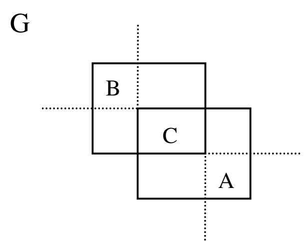

Figure 4.1

And we define *A, B, C* region according to p_1 and q_1. Algorithm DP first will disjoint partition $\overline{B}$ maximally such that each partition dominates *A* (suppose the size of this partition is *m*). Because only $\overline{B}$ can dominate *A*, *m* will be the upper bound of *dm(G)*.

Second, algorithm DP will assign nodes of *B* to these partitions such that there are maximum dominating sets of *G* (say *dm(G)*).

We prove this by contradiction. In the first step, let the size of disjoint partition be *m'*, where *m'* < *m*, This will obtain a larger domatic number *dm'(G)*, *dm'(G)* > *dm(G)*. In another, after the first step, the total nodes needed from *B* is: $\beta m'$ - $(q_1^x - p_1^x) + (q_1^y - p_1^y)$, where β is a constant only depending on region *B*, then we have :

$$\beta m' - (q_1^x - p_1^x) + (q_1^y - p_1^y)$$
$$< \beta m - (q_1^x - p_1^x) + (q_1^y - p_1^y)$$

Because nodes of region *B* are the same, i.e., *dm'(G)* ≤ *dm(G)*. It means that we need fewer nodes to generate larger dominating sets. A contradiction occurs.

For *#stages* = *n*, the previous *n-1* stages satisfy induction hypothesis, show in Figure 4.2 :

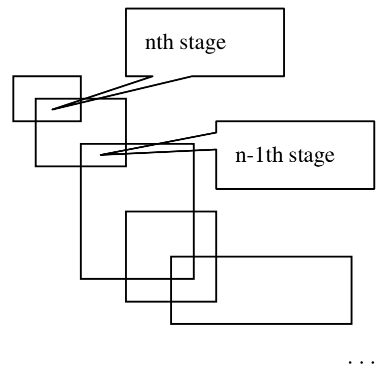

Figure 4.2

We can prove this case the same way as we do in the basis step.

It is easy to verify that Algorithm DP scans each vertex and each edge constant times. Therefore, the following Theorem can be obtained.

Theorem 4.1 *The time complexity of Algorithm DP is O(n+m).*

5 Conclusions

Domatic number problem is so essential and fundamental to graph theory and its applications. It has been known that the problem is NP-Hard on bipartite graphs. This paper has proposed a linear-time algorithm for the problem on bipartite permutation graphs. It is worthy to extend the result of this paper to the super classes of bipartite permutation graphs such as double convex bipartite graphs, convex bipartite graphs, and permutation graphs. Meanwhile, it is a practical issue to derive efficient approximation algorithms for bipartite graphs by modifying our algorithm. We are now working this issue.

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