遞迴圖上泛迴圈相關性質之研究

Pancyclic Properties of the WK-Recursive Networks

方覺非 [1,*] 馮士丞 [1] 黃玉潔 [2]
Jywe-Fei Fang Shr-Cheng Feng Yu-Chieh Huang

[1] 國立台中教育大學
數位內容科技系
Department of Digital Content and Technology
National Taichung University
* jffang@mail.sju.edu.tw

[2] 國立台中教育大學
資訊科學系
Department of Computer and Information Science
National Taichung University

Abstract

In this paper, we study the pancyclic properties of the WK-Recursive networks. We show that a WK-Recursive network with amplitude W and level L is vertex-pancyclic for $W \geq 6$. That is, each vertex on them resides in cycles of all lengths ranging from 3 to N, where N is the size of the interconnection network. On the other hand, we also investigate the m-edge-pancyclicity of the WK-Recursive network. We show that the WK-Recursive network is strictly $3\times2^{L-1}$-edge-pancyclic for $W \geq 7$ and $L \geq 1$. That is, each edge on them resides in cycles of all lengths ranging from $3\times2^{L-1}$ to N; and the value $3\times2^{L-1}$ reaches the lower bound of the problem.

Keyword: Pancyclicity, Interconnection networks, WK-Recursive networks.

1. Introduction

In massively parallel MIMD systems, the topology plays a crucial role in issues such as communication performance, hardware cost, potentialities for efficient applications and fault tolerant capabilities [9, 16]. A topology named *WK-Recursive network* has been proposed [24]. The topology has many attractive properties, such as high degree of regularity, symmetry and efficient communication. Particularly, for any specified number of degree, it can be expanded to an arbitrary size level without reconfiguring the edges. Because it demonstrates many attractive properties, researchers have devoted themselves to various issues of the WK-Recursive networks, such as broadcasting algorithms [12], topological properties [18, 14] and communication [8].

Paths and *cycles* are popular interconnection networks owing to their simple structures and low degree. Moreover, many parallel algorithms have been devised on them [16, 17, 19]. Many literatures have discussed how to embed cycles and paths into various topologies [3, 5, 16]. A cycle with length s is denoted by C_s, where $s \geq 3$. A graph is *Hamiltonian* if it embeds a *Hamiltonian cycle* that contains each vertex exactly once [7]. In other words, that a graph is Hamiltonian implies that it embeds the maximal cycle. However, in the *resource-allocated systems*, each vertex may be allocated with or without a resource [4, 10]. Thus, it makes sense to discuss how to join a specific pair of vertices with a *Hamiltonian path* in such systems. For example, let X and Y be two vertices in a resource-allocated system, where the former and the latter are assigned with an input device and an output device, respectively. If we find a Hamiltonian path joining the pair of vertices, we can utilize the whole system to perform the systolic algorithm on a linear array [19]. A graph is Hamiltonian-connected if there is a Hamiltonian path joining each pair of vertices. No wonder that there are many researchers discussing the *Hamiltonian-connectivity* of various interconnection networks [6, 20].

On the other hand, to execute a parallel program efficiently, the size of the allocated cycle must accord with the problem size of the program. Thus, many researchers study the problem of how to embed cycles of different sizes into an interconnection network. A graph is *pancyclic* if it embeds cycles of every length ranging from 3 to N, where N is the size of the graph [2]. A graph is *m-pancyclic* if it embeds cycles of every length ranging from m to N, where $3 \leq m \leq N$. Clearly, an m-pancyclic graph must be Hamiltonian. In a *heterogeneous computing system*, each vertex and each edge may be assigned with distinct computing power and distinct bandwidth, respectively [22]. Thus, it is meaningful to extend the pancyclicity to the *vertex-pancyclicity* and the *edge-pancyclicity* [1, 11, 15]. A graph is vertex-pancyclic (edge-pancyclic) if each vertex (edge) lies on cycles of every length ranging from 3 to N. Clearly, each edge-pancyclic graph must be vertex-pancyclic.

The concepts of the vertex-pancyclicity and the edge-pancyclicity are generalized to the *m-vertex-pancyclicity* and the *m-edge-pancyclicity* [21]. A graph is said to be m-vertex-pancyclic (m-edge-pancyclic) if each vertex (edge) lies on cycles of all lengths ranging from m to N. Obviously, every m_1-vertex-pancyclic (m_1-edge-pancyclic) graph must be m_2-vertex-pancyclic (m_2-edge-pancyclic), where $N \geq m_2 \geq m_1$. A graph is *strictly m-vertex-pancyclic* (*m-edge-pancyclic*) if it is not (m-1)-vertex-pancyclic ((m-1)-edge-pancyclic) but m-vertex-pancyclic (m-edge-pancyclic); that is, the value m reaches the lower bound of the problem. Clearly, every m-edge-pancyclic graph is m-vertex-pancyclic. A graph G with N vertices is *panconnected* if for each pair of distinct vertices X, Y and for any integer $d(X, Y) \leq l \leq N$-1, there exists a path of length l in G connecting X and Y, where $d(X, Y)$ is the distance between X and Y in G [6].

A WK-Recursive network with amplitude W and level L is denoted by a WK(W, L). Vecchia and Sanges have shown that a WK(W, L) is Hamiltonian for $W \geq 3$ [24]. Fernandes et al. have shown that a WK(W, L) is pancyclic for $W \geq 5$ [13]. However, to the best of our knowledge, there exists no article addressing the m-vertex-pancyclicity and the m-edge-pancyclicity of the WK(W, L). In this paper, we show that a WK(W, L) is vertex-pancyclic for $W \geq 6$. The WK(W, L) network is strictly 7-vertex-pancyclic, for $5 \geq W \geq 4$ and $L \geq 2$. On the other hand, we also prove that the WK(W, L) is strictly $3 \times 2^{L-1}$-edge-pancyclic for $W \geq 7$ and $L \geq 1$.

The rest of this paper is organized as follows. In Section 2, we present some notations and background that will be used throughout this paper. In Section 3, we study the pancyclicity and the m-vertex-pancyclicity of the WK-Recursive network. In Section 4, we investigate the m-edge-pancyclicity of the WK-Recursive network. Conclusions are given in Section 5.

2. Notations and background

A WK(W, L) can be recursively constructed. A WK(W, 0) is a vertex with W free edges. A WK(W, 1) is a W-vertex complete graph that is denoted by a K_W. Each vertex has one free edge and W-1 edges that are used for connecting to the other vertices. Clearly, a WK(W, 1) has W vertices and W free edges. A WK(W, H) consists of W copies of WK(W, H-1) as supervertices and the W supervertices are connected as a K_W, where $2 \leq H \leq L$. By induction, it is easy to see that a WK(W, L) has W^L vertices and W free edges. Consequently, for any given amplitude W, WK-Recursive networks can be expanded to any arbitrary level L without reconfiguring the edges. In Figure 1, the structures of a WK(4, 0), a WK(4, 1), a WK(4, 2) and a WK(4, 3) are illustrated.

The following addressing scheme for a WK(W, L) is described in [23]. After fixing an origin and an orientation (clockwise or counterclockwise), for each WK(W, 1) subnetwork, every vertex is labeled with an index digit $d_1 \in \{0, 1, \ldots, W\text{-}1\}$. Likewise, for each WK($W$, H) subnetwork, every WK(W, H-1) subnetwork is labeled with an index $d_H \in \{0, 1, \ldots, W\text{-}1\}$, where $2 \leq H \leq L$. Hence, each vertex of a WK(W, L) is labeled with an unique address $d_L d_{L-1} \ldots d_2 d_1$ as illustrated in Figure 1. Likewise, a subnetwork of a WK(W, L) can be represented by a string of L symbols over set $\{0, 1, \ldots, W\text{-}1\} \cup \{*\}$, where * is a “don't care” symbol. That is, each WK(W, H) subnetwork of a WK(W, L) can be denoted by $d_L d_{L-1} \ldots d_{H+1}(*)^H$, where $(*)^H$ represents H consecutive *'s. For example, in a WK(4, 3), 0** is the subnetwork $\{0d_2d_1 \mid 0 \leq d_2 \leq 3 \text{ and } 0 \leq d_1 \leq 3\}$.

For a subnetwork $d_L d_{L-1} \ldots d_{H+1}(*)^H$ in a WK(W, L), a vertex with address $d_L d_{L-1} \ldots d_{H+1}(d_H)^H$ is called a *corner vertex* of $d_L d_{L-1} \ldots d_{H+1}(*)^H$. For example, in a WK(4, 3), 000, 011, 022 and 033 are corner vertices of 0**. Specifically, the vertex $d_L d_{L-1} \ldots d_{H+1}(d_H)^H$ is called the *d_H-corner* of $d_L d_{L-1} \ldots d_{H+1}(*)^H$. For example, in a WK(4, 3), 033 is called 3-corner of 0**. In this paper, an edge within a WK(W, 1) subnetwork is called an *inner-cluster edge*.

Definition 1. The inner-cluster edges of vertex $d_L d_{L-1} \ldots d_2 d_1$ are defined as $(d_L d_{L-1} \ldots d_2 d_1, d_L d_{L-1} \ldots d_2 h)$, where $0 \leq h \leq W$-1 and $d_1 \neq h$.

For example, in a WK(5, 3), (002, 000), (002, 001), (002, 003) and (002, 004) are inner-cluster edges of the vertex 002. Clearly, each vertex has W-1 inner-cluster edges in a WK(W, L). An edge connecting two WK(W, H) subnetworks, where $1 \leq H \leq L$-1, is called an *inter-cluster edge* and specifically a *level-H edge*.

Definition 2. The level-H inter-cluster edge of vertex $d_L d_{L-1} \ldots d_{H+1}(d_H)^H$, where $d_{H+1} \neq d_H$, is defined as $(d_L d_{L-1} \ldots d_{H+1}(d_H)^H, d_L d_{L-1} \ldots d_H(d_{H+1})^H)$.

For example, in a WK(4, 3), (022, 200) is a level-2 edge. Observe that each vertex except the corner vertices $(d_L)^L$ has exactly one inter-cluster edge in a WK(W, L). Each corner vertex $(d_L)^L$ of a WK(W, L) has no inter-cluster edge but a free edge.

In this paper, the *outline graph* of a WK(W, L), denoted by an OG(WK(W, L)), is to take each WK(W, 1) subnetwork as a supervertex. As stated before, a

WK(W, L) can be constructed recursively. If each WK(W, 1) subnetwork of a WK(W, L) is taken as a supervertex, the WK(W, L) will be transformed to a WK(W, L-1). Moreover, each original level-1 inter-cluster edge will be an inner-cluster edge in the WK(W, L-1); and each original level-J inter-cluster edge will be a level-(J-1) inter-cluster edge in the WK(W, L-1), where $L-1 \geq J \geq 2$. We have the following proposition.

Proposition 1. An OG(WK(W, L)) is a WK(W, L-1).

As illustrated in Figure 2, the OG(WK(4, 3)) is a WK(4, 2). Because an OG(WK(W, L)) is a WK(W, L-1); and each vertex of the WK(W, L-1) has W-1 inner-cluster edges, thus we have

Proposition 2. In each WK(W, 1) subnetwork of a WK(W, L), there exist W-1 level-1 edges and at most one higher level edge.

3. The Pancyclicity and the Vertex-Pancyclicity of a WK-Recursive Network

In this section, we will discuss the pancyclicity and the vertex-pancyclicity of the WK-Recursive network. Suppose that vertices $U_1, U_2, \ldots, U_W$ locate in a common WK(W, 1) subnetwork. Because a WK(W, 1) subnetwork is a K_W, (U_1, U_W) forms a path of length 1. The vertex U_j is called an *appending vertex*, where $2 \leq j \leq W-1$. Clearly, appending vertices can be appended one by one to the path. That is, ($U_1, U_2, \ldots, U_i, U_W$) forms a path of length i, where $1 \leq i \leq W-1$. Recall that OG(WK(W, L)) is obtained from WK(W, L) by taking each WK(W, 1) subnetwork as a supervertex. Suppose that there is a cycle of length l, denoted by ($V_0^*, V_1^*, V_2^*, \ldots, V_{l-1}^*$), in the OG(WK(W, L)) as illustrated in Figure 3. Clearly, there exists an inter-cluster edge connecting consecutive two WK(W, 1) supervertices in the cycle.

Since each vertex has one inter-cluster edge at most, we can find two vertices S_i and D_i in each supervertex V_i^*, where $0 \leq i \leq l-1$, such that they are incident to the inter-cluster edges connecting to $V_{(i-1) \bmod l}^*$ and $V_{(i+1) \bmod l}^*$, respectively. Obviously, ($S_0, D_0, S_1, D_1, \ldots, S_{l-1}, D_{l-1}$) forms a cycle of length $2l$. In each supervertex V_i^*, there are W-2 appending vertices. Totally, there exist (W-2)l appending vertices. Thus, we have the following lemma.

Lemma 1. If there exists a cycle of length l in the OG(WK(W, L)), a WK(W, L) embeds cycles of all lengths ranging from $2l$ to Wl.

Theorem 2. A WK(W, L) is pancyclic for $W \geq 5$.

Proof. We will prove the theorem by induction on L.

For L = 1, a WK(W, 1) is a K_W. Clearly, it embeds cycles of all lengths ranging from 3 to W, where $W \geq 5$.

Hypothesis: Suppose that a WK(W, k) is pancyclic.

Induction Step: Recall that OG(WK(W, k+1)) is a WK(W, k) network. By the hypothesis, we know that a WK(W, k) embeds cycles of all lengths ranging from 3 to W^k. Thus, by Lemma 1, a WK(W, k+1) can embed

$\{C_s|\ 6 \leq s \leq 3W\}$ (for 3 WK(W, 1) supervertices)
$\cup \{C_s|\ 8 \leq s \leq 4W\}$ (for 4 WK(W, 1) supervertices)
…,
$\cup \{C_s|\ 2n \leq s \leq nW\}$ (for n WK(W, 1) supervertices)
$\cup \{C_s|\ 2(n+1) \leq s \leq (n+1)W\}$ (for n+1 WK(W, 1) supervertices)
…,
$\cup \{C_s|\ 2W^k \leq s \leq W^kW = W^{k+1}\}$ (for W^k WK(W, 1) supervertices).

Clearly, nW is always greater than $2(n+1)$ for $W \geq 5$ and $n \geq 3$. Thus, a WK(W, k+1) can embed $\{C_s|\ 6 \leq s \leq W^{k+1}\}$. By the recursive structure of the WK-Recursive Network, the WK(W, 1) is a subgraph of a WK(W, k+1) for $k \geq 1$. Thus, we know that a WK(W, k+1) can embed $\{C_s|\ 3 \leq s \leq W\} \cup \{C_s|\ 6 \leq s \leq W^{k+1}\} = \{C_s|\ 3 \leq s \leq W^{k+1}\}$, where $W \geq 5$.

This extends the induction and completes the proof. Q. E. D.

Although, in fact, the above theorem has been shown by Fernandes et al. [13], our proof is much easier and clearer. Moreover, we will discuss the m-vertex-pancyclicity and the m-edge-pancyclicity of a WK(W, L) upon the above discussion.

In the following, we investigate the m-pancyclicity of a WK(4, L). Obviously, a WK(4, 1) embeds C_3 and C_4. By Lemma 1, a WK(4, 2) can embed $\{C_s|\ 6 \leq s \leq 12\} \cup \{C_s|\ 8 \leq s \leq 16\} = \{C_s|\ 6 \leq s \leq 16\}$. Suppose that a WK(4, k) can embed $\{C_s|\ 6 \leq s \leq 4^k\}$ for $k \geq 3$. Likewise, a WK(4, k+1) can embed $\{C_s|\ 12 \leq s \leq 24\} \cup \{C_s|\ 14 \leq s \leq 28\} \cup \ldots \cup \{C_s|\ 2(4^k-1) \leq s \leq 4(4^k-1)\} \cup \{C_s|\ 2\times4^k \leq s \leq 4\times4^k = 4^{k+1}\} = \{C_s|\ 12 \leq s \leq 4^{k+1}\}$. By the recursive structure of a WK(W, L), the WK(4, 2) is a subgraph of a WK(4, L) for $L > 2$. Thus, a WK(4, k+1) can embed $\{C_s|\ 6 \leq s \leq 16\} \cup \{C_s|\ 12 \leq s \leq 4^{k+1}\} = \{C_s|\ 6 \leq s \leq 4^{k+1}\}$. Thus, we have

Lemma 3. A WK(4, L) is 6-pancyclic, where $L \geq 2$.

Then, the m-pancyclicity of a WK(3, L) is studied. Because a WK(3, 1) is nothing but a C_3, a WK(3, 2) can embed $\{C_s|\ 6 \leq s \leq 9\}$ by Lemma 1. We have

Corollary 4. A WK(3, 2) is 6-pancyclic.

A WK(3, 3) can embed $\{C_s|\ 12 \leq s \leq 27\}$. Similarly, if a WK(3, k) can embed $\{C_s|\ 12 \leq s \leq 3^k\}$, where $k \geq 3$, a WK(3, k+1) can embed $\{C_s|\ 12 \leq s \leq 27\} \cup \{C_s|\ 24 \leq s \leq 3^{k+1}\} = \{C_s|\ 12 \leq s \leq 3^{k+1}\}$. Thus, we have

Lemma 5. A WK(3, L) is 12-pancyclic for $L \geq 3$.

In the following, we investigate the vertex-pancyclicity and the *m*-vertex-pancyclicity of a WK(W, L).

Lemma 6. If each WK(W, 1) supervertex resides in a cycle of length l in the *OG*(WK(W, L)), each vertex of a WK(W, L) resides in cycles of all lengths ranging from $2l+1$ to Wl.

Proof. Suppose that an arbitrary vertex X of a WK(W, L) reside in the supervertex V_0* of the *OG*(WK(W, L)). By the hypothesis, there exists a cycle, denoted by (V_0*, V_1*, V_2*, …, V_{l-1}*), of length l containing V_0* in the *OG*(WK(W, L)). As illustrated in Figure 3, there exists an inter-cluster edge connecting consecutive two WK(W, 1) supervertices in the cycle.

Since each vertex has one inter-cluster edge at most, we can find two vertices S_i and D_i in each supervertex V_i*, where $0 \le i \le l-1$, such that they are incident to the inter-cluster edges connecting to $V_{(i-1) \bmod l}$* and $V_{(i+1) \bmod l}$*, respectively. Obviously, (S_0, D_0, S_1, D_1, …, S_{l-1}, D_{l-1}) forms a cycle of length $2l$.

Case 1: X is not S_0 and X is not D_0. Clearly, (S_0, X, D_0, S_1, D_1, …, S_{l-1}, D_{l-1}) forms a cycle of length $2l+1$. Totally, there are $(W-2)l-1$ appending vertices. Thus, X resides in cycles of all lengths ranging from $2l+1$ to Wl.

Case 2: X is S_0 or X is D_0. Clearly, X resides in cycles of all lengths ranging from $2l$ to Wl. Q. E. D

Theorem 7. A WK(W, L) is vertex-pancyclic for $W \ge 6$.

Proof. We will prove the theorem by induction on L.

For L =1, a WK(W, 1) is a K_W. Clearly, it's vertex-pancyclic.

Hypothesis: Suppose that a WK(W, k) is vertex-pancyclic, where $W \ge 6$.

Induction Step: Because the *OG*(WK(W, k+1)) is a WK(W, k). By the hypothesis, we know that each supervertex of the *OG*(WK(W, k+1)) resides in cycles of all lengths ranging from 3 to W^k. From the Lemma 6, we know that each vertex of a WK(W, k+1) resides in

$\{C_s | 7 \le s \le 3W\}$ (for 3 WK(W, 1) supervertices)
$\cup \{C_s | 9 \le s \le 4W\}$ (for 4 WK(W, 1) supervertices)
…,
$\cup \{C_s | 2n+1 \le s \le nW\}$ (for n WK(W, 1) supervertices)
$\cup \{C_s | 2n+3 \le s \le (n+1)W\}$ (for n+1 WK(W, 1) supervertices)
…,
$\cup \{C_s | 2W^k+1 \le s \le W^kW = W^{k+1}\}$ (for W^k WK(W, 1) supervertices).

For $W \ge 6$ and $n \ge 3$, nW is always greater than $2n+3$. Thus, each vertex of a WK(W, k+1) resides in cycles of all lengths ranging from 7 to W^{k+1}. By the recursive structure of the WK-Recursive Network, each vertex of a WK(W, k+1) must reside in a WK(W, 1) subnetwork. Thus, each vertex of a WK(W, k+1) resides in $\{C_s | 3 \le s \le W\} \cup \{C_s | 7 \le s \le W^{k+1}\} = \{C_s | 3 \le s \le W^{k+1}\}$ for $W \ge 6$. Q. E. D.

Then, we investigate the vertex-pancyclicity of a WK(4, L) and a WK(5, L). Clearly, a WK(4, 1) and a WK(5, 1) are vertex-pancyclic for L = 1. That is, each vertex of a WK(4, 1) (WK(5, 1)) resides in $\{C_s | 3 \le s \le 4\}$ ($\{C_s | 3 \le s \le 5\}$). From Lemma 6, we know that each vertex of a WK(4, 2) (WK(5, 2)) resides in $\{C_s | 7 \le s \le 16\}$ ($\{C_s | 7 \le s \le 25\}$). Thus, a WK(4, 2) and a WK(5, 2) are 7-vertex-pancyclic. Suppose that a WK(4, k) and a WK(5, k) are 7-vertex-pancyclic for $k \ge 2$. By the recursive structure of the WK(W, L), each vertex of a WK(W, k+1) must reside in a WK(W, 2) subnetwork for $k > 2$. According to Lemma 6, each vertex of a WK(4, k+1) (WK(5, k+1)) resides in $\{C_s | 7 \le s \le 16\} \cup \{C_s | 15 \le s \le 4^{k+1}\} = \{C_s | 7 \le s \le 4^{k+1}\}$ ($\{C_s | 7 \le s \le 25\} \cup \{C_s | 15 \le s \le 5^{k+1}\} = \{C_s | 7 \le s \le 5^{k+1}\}$). Thus, we have

Lemma 8. A WK(4, L) and a WK(5, L) are 7-vertex-pancyclic, where $L \ge 2$.

In a WK(4, L) (WK(5, L)), a corner vertex $(d_L)^L$ has no inter-cluster edge, where $0 \le d_L \le 3$ ($0 \le d_L \le 4$) and $L \ge 2$. Thus, the corner vertex cannot reside in a C_6.

Corollary 9. A WK(4, L) and a WK(5, L) are strictly 7-vertex-pancyclic, where $L \ge 2$.

A WK(3, 1) is nothing but a C_3. From Lemma 6, each vertex of a WK(3, 2) resides in $\{C_s | 7 \le s \le 9\}$. Each vertex of a WK(3, 3) resides in $\{C_s | 15 \le s \le 27\}$. Each vertex of a WK(3, 4) resides in $\{C_s | 31 \le s \le 81\}$. If each vertex of a WK(3, k) resides in $\{C_s | 31 \le s \le 3^k\}$ for $k \ge 4$, each vertex of a WK(3, k+1) resides in $\{C_s | 63 \le s \le 3^{k+1}\}$. By the recursive structure of the WK-Recursive Network, each vertex of a WK(W, k+1) must reside in a WK(W, k) subnetwork. Clearly, 3^k is always greater than 63 for $k \ge 4$. Thus, each vertex of a WK(3, k+1) resides in $\{C_s | 31 \le s \le 3^k\} \cup \{C_s | 63 \le s \le 3^{k+1}\} = \{C_s | 31 \le s \le 3^{k+1}\}$ for $k \ge 4$. Thus we have

Lemma 10. A WK(3, L) is 31-vertex-pancyclic for $L \ge 4$.

4. The Edge-Pancyclicity of a WK-Recursive Network

In this section, we investigate the edge-pancyclicity of the WK-Recursive network. To study the edge-pancyclicity of a WK-Recursive network, the following lemmas are required.

Lemma 11. If each inner-cluster edge of the *OG*(WK(W, L)) resides in a cycle of length l, each inner-cluster edge of a WK(W, L) resides in cycles of all lengths ranging from $2l+2$ to Wl, where $W \ge 4$.

Proof. In a WK(W, L), let (X, Y) be an arbitrary inner-cluster edge residing in an arbitrary WK(W, 1) supervertex V_1*. By Proposition 2, because $W \geq 4$, there exists a vertex S_1 residing in the V_1* such that $S_1 \neq X$ and $S_1 \neq Y$; and S_1 is incident to a level-1 edge (V_0*, V_1*) of the WK(W, L). By the hypothesis, we know that the inner-cluster edge (V_0*, V_1*) of the OG(WK(W, L)) resides in a cycle (V_0*, V_1*, …, V_{l-1}*) of length l, where $3 \leq l \leq W^{L-1}$, as illustrated in Figure 3. Let D_1 be the vertex that resides in V_1* and is incident to the next edge (V_1*, V_2*) of the cycle.

Case 1: $X \neq D_1$ and $Y \neq D_1$. Clearly, (S_0, D_0, S_1, X, Y, D_1, S_2, D_2, …, S_{l-1}, D_{l-1}) forms a cycle of length $2l+2$. Totally, there are $(W-2)l-2$ appending vertices. Thus, (X, Y) resides in cycles of all lengths ranging from $2l+2$ to Wl.

Case 2: $X = D_1$ or $Y = D_1$. Without loss of generality, let $Y = D_1$. Clearly, (S_0, D_0, S_1, X, Y (i.e., D_1), S_2, D_2, …, S_{l-1}, D_{l-1}) forms a cycle of length $2l+1$. Totally, there are $(W-2)l-1$ appending vertices. Thus, (X, Y) resides in cycles of all lengths ranging from $2l+1$ to Wl. Q. E. D.

Lemma 12. Each inner-cluster edge of a WK(W, L), where $W \geq 7$, resides in cycles of all lengths ranging from 3 to W^L.

Proof. We will prove the lemma by induction on L.

For L =1, a WK(W, 1) is a K_W. Clearly, the lemma is true.

Hypothesis: Suppose that each inner-cluster edge of a WK(W, k), where $W \geq 7$, resides in cycles of all lengths ranging from 3 to W^k.

Induction Step: The OG(WK(W, k+1)) is a WK(W, k). By the hypothesis and Lemma 11, we know that each inner-cluster edge of a WK(W, k+1) resides in

$\{C_s | 8 \leq s \leq 3W\}$ (for 3 WK(W, 1) supervertices)
$\cup \{C_s | 10 \leq s \leq 4W\}$ (for 4 WK(W, 1) supervertices)
…,
$\cup \{C_s | 2n+2 \leq s \leq nW\}$ (for n WK(W, 1) supervertices)
$\cup \{C_s | 2n+4 \leq s \leq (n+1)W\}$ (for n+1 WK(W, 1) supervertices)
…,
$\cup \{C_s | 2W^k+2 \leq s \leq W^kW = W^{k+1}\}$ (for W^k WK(W, 1) supervertices).

For $W \geq 7$ and $n \geq 3$, nW is always greater than $2n+4$. Thus, each inner-cluster edge of a WK(W, k+1) resides in cycles of all lengths ranging from 8 to W^{k+1}. By the recursive structure of the WK-Recursive Network, each inner-cluster edge of a WK(W, k+1) must reside in a WK(W, 1) subnetwork for $k \geq 1$. Thus, each inner-cluster edge of a WK(W, k+1) resides in $\{C_s | 3 \leq s \leq 7\} \cup \{C_s | 8 \leq s \leq W^{k+1}\} = \{C_s | 3 \leq s \leq W^{k+1}\}$, where $W \geq 7$. This extends the induction and completes the proof. Q. E. D.

Clearly, each inner-cluster edge of a WK(W, 1), where $5 \leq W \leq 6$, resides in cycles of all lengths ranging from 3 to W. By Lemma 11, each inner-cluster edge of a WK(W, 2), where $5 \leq W \leq 6$, resides in $\{C_s | 3 \leq s \leq W$ or $8 \leq s \leq W^2\}$. Suppose that each inner-cluster edge of a WK(W, k), where $5 \leq W \leq 6$, resides in $\{C_s | 3 \leq s \leq W$ or $8 \leq s \leq W^k\}$. By Lemma 11, we know that each inner-cluster edge of a WK(W, k+1), where $5 \leq W \leq 6$, resides in $\{C_s | 3 \leq s \leq W$ or $8 \leq s \leq W^2$ or $18 \leq s \leq W^{k+1}\}$. For $5 \leq W \leq 6$, W^2 is always greater than 18. Thus, we have

Corollary 13. Each inner-cluster edge of a WK(W, L), where $5 \leq W \leq 6$, resides in $\{C_s | 3 \leq s \leq W$ or $8 \leq s \leq W^L\}$.

Likewise, we have the following corollary.

Corollary 14. Each inner-cluster edge of a WK(4, 2) resides in $\{C_s | 3 \leq s \leq 4$ or $8 \leq s \leq 16\}$.

Each inner-cluster edge of a WK(4, 3) resides in $\{C_s | 3 \leq s \leq 4$ or $8 \leq s \leq 16$ or $18 \leq s \leq 64\}$. Suppose that each inner-cluster edge of a WK(4, k) resides in $\{C_s | 3 \leq s \leq 4$ or $8 \leq s \leq 16$ or $18 \leq s \leq 4^k\}$ for $k \geq 3$. By Lemma 11, each inner-cluster edge of a WK(4, k+1) resides in $\{C_s | 3 \leq s \leq 4$ or $8 \leq s \leq 16$ or $18 \leq s \leq 4^k$ or $38 \leq s \leq 4^{k+1}\}$. Clearly, for $k \geq 3$, 4^k is always greater than 38. Thus, we have

Corollary 15. Each inner-cluster edge of a WK(4, L) resides in $\{C_s | 3 \leq s \leq 4$ or $8 \leq s \leq 16$ or $18 \leq s \leq 4^L\}$ for $L \geq 3$.

Lemma 16. There exist paths of all lengths ranging from 2^L-1 to W^L-1, between each pair of corner vertices of a WK(W, L), where $W \geq 4$.

Proof. We will prove the lemma by induction on L.

For L = 1, a WK(W, 1) is a K_W. Clearly, there exist paths of all lengths ranging from 2^L-1(i.e., 1) to W^L-1 (i.e., W-1), between each pair of corner vertices of a WK(W, 1).

Hypothesis: Suppose that there exist paths of all lengths ranging from 2^k-1 to W^k-1, between each pair of corner vertices of a WK(W, k), where $W \geq 4$.

Induction Step: The OG(WK(W, k+1)) is a WK(W, k). Let (P, Q) be a pair of corner vertices of a WK(W, k+1). The WK(W, 1) supervertex that P(Q) resides in is denoted by V_P* (V_Q*). By the hypothesis, we know that there exist a path (V_0*, V_1*, …, V_{l-1}*) of length l-1, where $2^k \leq l \leq W^k$, V_0* = V_P* and V_{l-1}* = V_Q*. There exist paths of all lengths ranging from $2\times2^k-1 = 2^{k+1}-1$ to $W\times W^k-1 = W^{k+1}-1$, between P and Q. This extends the induction and completes the proof. Q. E. D.

If each WK(W, 1) subnetwork of a WK(W, L) is taken as a supervertex, the WK(W, L) will be transformed to a WK(W, L-1). Moreover, each original level-1 inter-cluster edge will be an inner-cluster edge in the WK(W, L-1); and each original level-J inter-cluster edge will be a level-(J-1) inter-cluster edge in the WK(W, L-1), where L-1 ≥ J ≥ 2. As stated before, a WK(W, L) can be constructed recursively. By induction, it is easy to see that if each WK(W, H) subnetwork of a WK(W, L) is taken as a supervertex, the WK(W, L) will be transformed to a WK(W, L-H); the original level-H inter-cluster edge of the WK(W, L) will be transformed as an inner-cluster edge of the WK(W, L-H); and the original level-J inter-cluster edge of the WK(W, L) will be transformed as a level-(J-H) inter-cluster edge of the WK(W, L-H), where L-1 ≥ J > H.

Lemma 17. If each inner-cluster edge of a WK(W, L-H) resides in a cycle of length l, each level-H inter-cluster edge of a WK(W, L) resides in cycles of all lengths ranging from $2^H \times l$ to $W^H \times l$. Q. E. D

Proof: By hypothesis, we know that each inner-cluster edge of a WK(W, L-H) resides in a cycle of length l. Thus, in the corresponding WK(W, L), each level-H inter-cluster edge resides in the cycles through l WK(W, H) subnetworks and l level-H inter-cluster edges. From Lemma 16, we know that each level-H inter-cluster edge of a WK(W, L) resides in cycles of all lengths ranging from $(2^H\text{-}1)\times l + l = 2^H \times l$ to $(W^H\text{-}1)\times l + l = W^H \times l$. Q. E. D.

Lemma 18. A level-H inter-cluster edge of a WK(W, L), where $W \geq 7$ and $H \geq 1$, resides in cycles of all lengths ranging from 3×2^H to W^L.

Proof. By Lemma 12, each inner-cluster edge of the WK(W, L-H) resides in cycles of all lengths ranging from 3 to $W^{L\text{-}H}$, where $W \geq 7$. From Lemma 17, we know that each level-H inter-cluster edge of a WK(W, L) resides in $\{C_s|\ 3\times 2^H \leq s \leq 3\times W^H\} \cup \{C_s|\ 4\times 2^H \leq s \leq 4\times W^H\} \cup \ldots \cup \{C_s|\ (W^{L\text{-}H}-1)\times 2^H \leq s \leq (W^{L\text{-}H}-1)\times W^H\} \cup \{C_s|\ W^{L\text{-}H}\times 2^H \leq s \leq W^{L\text{-}H}\times W^H = W^L\} = \{C_s|\ 3\times 2^H \leq s \leq W^L\}$. That is, each level-$H$ inter-cluster edge of a WK(W, L) resides in cycles of all lengths ranging from 3×2^H to W^L, where $W \geq 7$ and $H \geq 1$. Q. E. D.

In this paper, the shortest path between X and Y is denoted by $X \Rightarrow^S Y$; and an edge connecting U and V is denoted by $U \rightarrow V$. The highest level of the inter-cluster edges of a WK(W, L) is L-1. Consider an level-(L-1) inter-cluster edge $(d_a(d_b)^{L\text{-}1}, d_b(d_a)^{L\text{-}1})$ of a WK(W, L), where $d_a \neq d_b$. By the structure of a WK(W, L), $d_a(*)^{L\text{-}1}$ and $d_b(*)^{L\text{-}1}$ are connected by only one edge $(d_a(d_b)^{L\text{-}1}, d_b(d_a)^{L\text{-}1})$. Thus, the shortest cycle embedding $(d_a(d_b)^{L\text{-}1}, d_b(d_a)^{L\text{-}1})$ is $(d_a(d_b)^{L\text{-}1}\rightarrow d_b(d_a)^{L\text{-}1} \Rightarrow^S d_b(d_c)^{L\text{-}1}\rightarrow d_c(d_b)^{L\text{-}1} \Rightarrow^S d_c(d_a)^{L\text{-}1} \rightarrow d_a(d_c)^{L\text{-}1} \Rightarrow^S d_a(d_b)^{L\text{-}1})$ where d_a, d_b and d_c are distinct digits. The distance between $d_b(d_a)^{L\text{-}1}$ and $d_b(d_c)^{L\text{-}1}$ ($d_c(d_b)^{L\text{-}1}$ and $d_c(d_a)^{L\text{-}1}$, $d_a(d_c)^{L\text{-}1}$ and $d_a(d_b)^{L\text{-}1}$) is $2^{L\text{-}1}$-1 [8]. The total length of the cycle is $3(2^{L\text{-}1}\text{-}1)+3 = 3\times 2^{L\text{-}1}$. Thus, we have

Lemma 19. The length of the shortest cycle containing a level-(L-1) inter-cluster edge of a WK(W, L) is $3\times 2^{L\text{-}1}$.

Combining Lemma 12, Lemma 18 and Lemma 19, we have

Theorem 20. A WK(W, L) is strictly $3\times 2^{L\text{-}1}$-edge-pancyclic, where $W \geq 7$ and $L \geq 1$.

Then, we investigate the m-edge-pancyclicity of a WK(5, L) and a WK(6, L). According to Corollary 13 and Lemma 17, we can derive that each level-H inter-cluster edge of a WK(5, L) resides in $\{C_s|\ 3\times 2^H \leq s \leq 5^L\}$ for $L \geq 3$. Clearly, the highest level of the inter-cluster edges of a WK(5, L) is L-1. From Corollary 13 and Lemma 19, we have the following lemmas:

Lemma 21. A WK(5, 2) is 8-edge-pancyclic.

Lemma 22. A WK(5, L) is strictly $3\times 2^{L\text{-}1}$-edge-pancyclic for $L \geq 3$.

Like the above discussion, we can derive the following lemmas:

Lemma 23. A WK(6, 2) is 8-edge-pancyclic.

Lemma 24. A WK(6, L) is strictly $3\times 2^{L\text{-}1}$-edge-pancyclic for $L \geq 3$.

Then, we investigate the m-edge-pancyclicity of a WK(4, L). By Corollary 15 and Lemma 17, we can derive that each level-H inter-cluster edge of a WK(4, L) resides in $\{C_s|\ 3\times 2^H \leq s \leq 4^L\}$ for $L \geq 4$. Clearly, the highest level of the inter-cluster edges of a WK(4, L) is L-1. From Corollary 14, Corollary 15 and Lemma 19, we have the following lemmas:

Lemma 25. A WK(4, 2) is 8-edge-pancyclic.

Lemma 26. A WK(4, 3) is 18-edge-pancyclic.

Lemma 27. A WK(4, L) is strictly $3\times 2^{L\text{-}1}$-edge-pancyclic for $L \geq 4$.

Consider the edge $((0)^{L\text{-}1}1, (0)^{L\text{-}1}2)$ of a WK(3, L), where $L \geq 2$. Clearly, the edge cannot be contained in a Hamiltonian cycle. As illustrated in Figure 4, edge (01, 02) cannot reside in a Hamiltonian cycle of a WK(3, 2). Thus, a WK(3, L) is not m-edge-pancyclic for $L \geq 2$.

5. Conclusions

In this paper, we have shown that a WK-Recursive network with amplitude W and level L is vertex-pancyclic for $W \geq 6$. The WK-Recursive network is proved to be strictly 7-vertex-pancyclic, where $5 \leq W \leq 6$ and $L \geq 2$. On the other hand, we also investigate the m-edge-pancyclicity of the WK-Recursive network. We show that the WK-Recursive network is strictly $3\times 2^{L\text{-}1}$-edge-pancyclic for $W \geq 7$ and $L \geq 1$. However, the panconnected problem of the WK-Recursive network is still open.

Acknowledgments

This work was supported in part by the National Science Council of the Republic of China under the contract number: NSC95-2221-E-142-003.

REFERENCE

[1] B. Alspach and D. Hare, "Edge-pancyclic block-intersection graphs", Discrete Math, 97 (1–3), pp.17–24, 1991.

[2] T. Araki and Y. Shibata, "Pancyclicity of recursive circulant graphs", Information Processing Letters, 81(4), pp.187-190, 2002.

[3] A. Auletta, A. Rescigno, and V. Scarano, "Embedding graphs onto the supercube", IEEE Trans. Comput, 44 (4), pp.593-597, 1995.

[4] M.M. Bae, "Resource placement in torus-based networks", IEEE Transactions on Comput, 46(10), 1083-1092, 1997.

[5] H.Y. Chang and R.J. Chen, "Embedding cycles in IEH graphs", Inform. Process. Lett, 64, pp.23-27, 1997.

[6] J. M. Chang, J. S.Yang, J. S. Yang, Y. L. Wang and Y. Cheng, "Panconnectivity, Fault-Tolerant Hamiltonicity and Hamiltonian-Connectivity in Alternating Group Graphs", Networks, 44(4), pp.302-310, 2004.

[7] G. Chartrand and O. R. Oellermann, "Applied and Algorithmic Graph Theory", McGraw-Hill, New York, 1993.

[8] G.H. Chen and D.R. Duh, "Topological properties, communication, and computation on WK-recursive networks", Networks, 24(6), pp.303-317, 1994.

[9] G.H. Chen, J.S. Fu and J.F. Fang, "Hypercomplete: A pancyclic recursive topology for large-scale distributed multicomputer systems", Network, 35(1), pp.56-69, 2000.

[10] H.L. Chen and N.F. Tzeng, "Efficient resource placement in hypercubes using multiple-adjacency codes", IEEE Trans. Comput. 43(1), pp.23-33, 1994.

[11] J. Fan, X. Lin and X. Jia, "Node-pancyclicity and edge-pancyclicity of crossed cubes", Information Processing Letters, 93(3), pp.133-138, 2005.

[12] J.F. Fang, G.J. Lai, Y.C. Liu and S.T. Fang, "A novel broadcasting scheme for WK-recursive networks", in Proceedings of 2003 IEEE Pacific Rim Conference on Communications, Computers and signal Processing, 2, pp.1028-1031, 2003.

[13] R. Fernandes, D.K. Friesen and A. Kanevsky, "Embedding rings in recursive networks", Proceedings of the International Parallel Processing Symposium, pp.273-280, 1994.

[14] J.S. Fu, "Hamiltonicity of the WK-recursive network with and without faulty nodes", IEEE Transactions on Parallel and Distributed Systems, 16(9), pp.853-865, 2005.

[15] A. Hobbs, "The square of a block is vertex pancyclic", J. Combin. Theory Ser. B, 20(1), pp.1-4, 1976.

[16] F.T. Leighton, "Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes, Mogran Kaufmann", California, 1992.

[17] Y.C. Lin, "On balancing sorting on a linear array", IEEE Transactions on Parallel and Distributed Systems, 4(5), pp.566-571, 1993.

[18] A.I. Mahdaly, H.T. Mouftah and N.N. Hanna, "Topological properties of WK-recursive networks", Proceed. Second IEEE Workshop on Future Trends of Distributed Computing Systems, pp.374-380, 1990.

[19] D.R. O'Hallaron, "Uniform approach for solving some classical problems on a linear array", IEEE Transactions on Parallel and Distributed Systems, 2(2), pp.236-241, 1991.

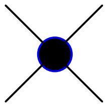
[20] C.D. Park and K.Y. Chwa, "Hamiltonian properties on the class of hypercube-like networks", Information Processing Letters, 91(1), pp.11-17, 2004.

[21] C. Peters and L. Volkmann, "Vertex 6-pancyclic in-tournaments", Discrete Math, 285(1–3), pp.227–238, 2004.

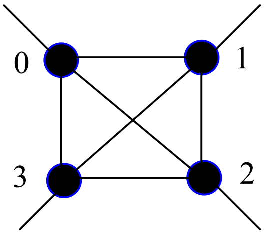
[22] B. Ucar, C. Aykanat, K. Kaya and M. Ikinci, "Task assignment in heterogeneous computing systems", Journal of Parallel and Distributed Computing, 66(1) , pp.32-46, 2006.

[23] G.D. Vecchia and C. Sanges, "Recursively scalable network for message passing architecture", Proceed. Int. Conf. Parallel Processing and Applications, 1, pp.33-40, 1987.

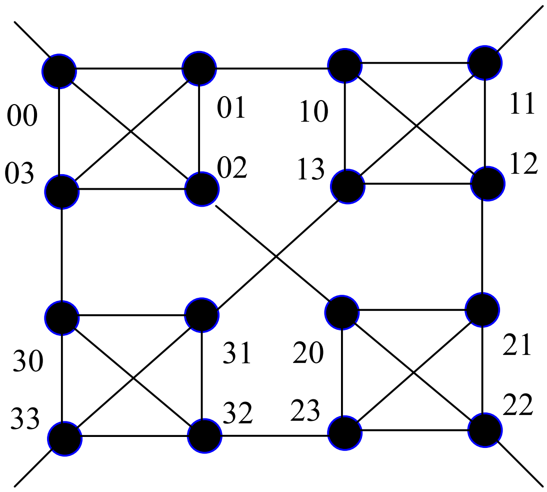
[24] G.D. Vecchia and C. Sanges, "A recursively scalable network VLSI implementation", Future Generat. Comput. Syst, 4(3), pp.235-243, 1988.



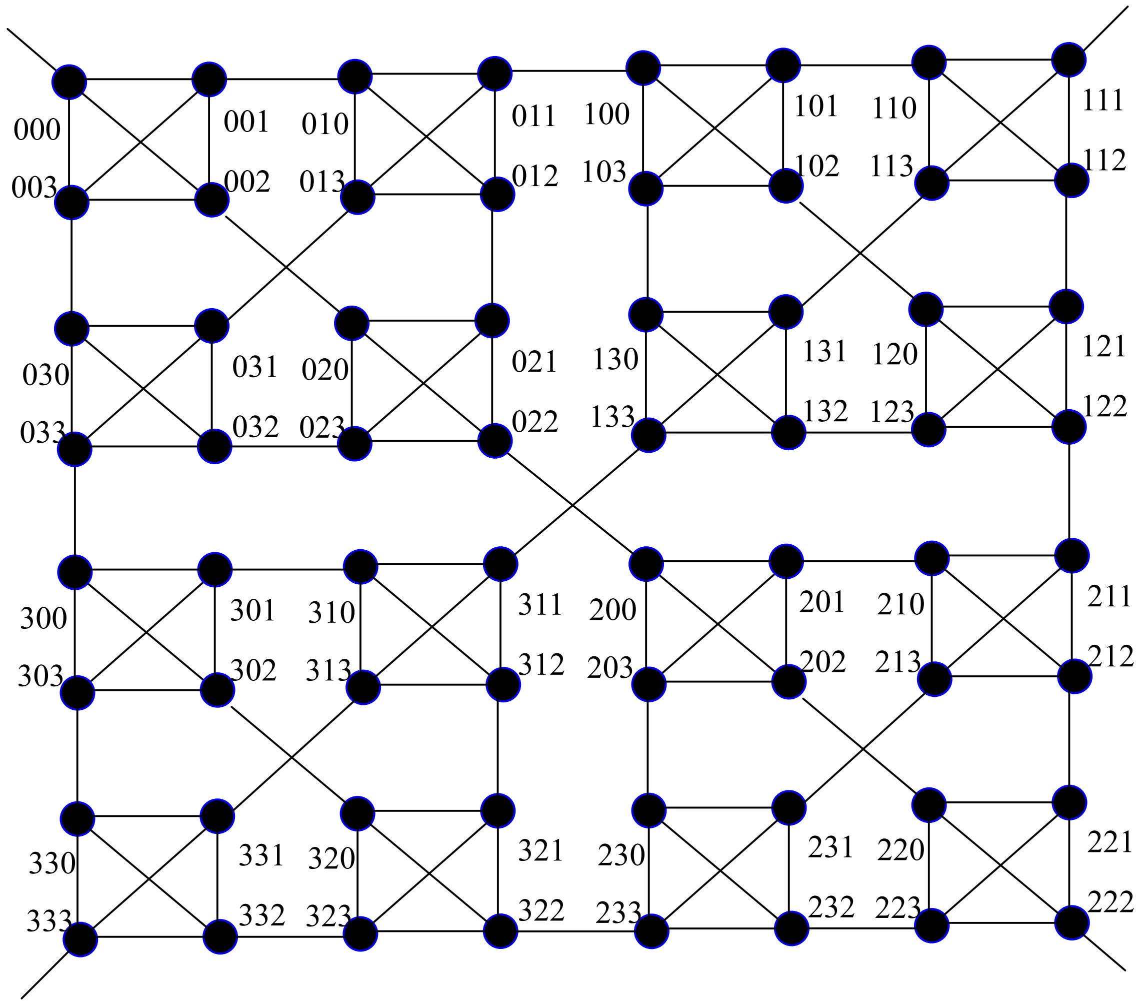
(a) A WK(4, 0).



(b) A WK(4, 1).



(c) A WK(4, 2).



(d) A WK(4, 3).

Figure 1. The structures of a WK(4, 0), a WK(4, 1), a WK(4, 2) and a WK(4, 3).

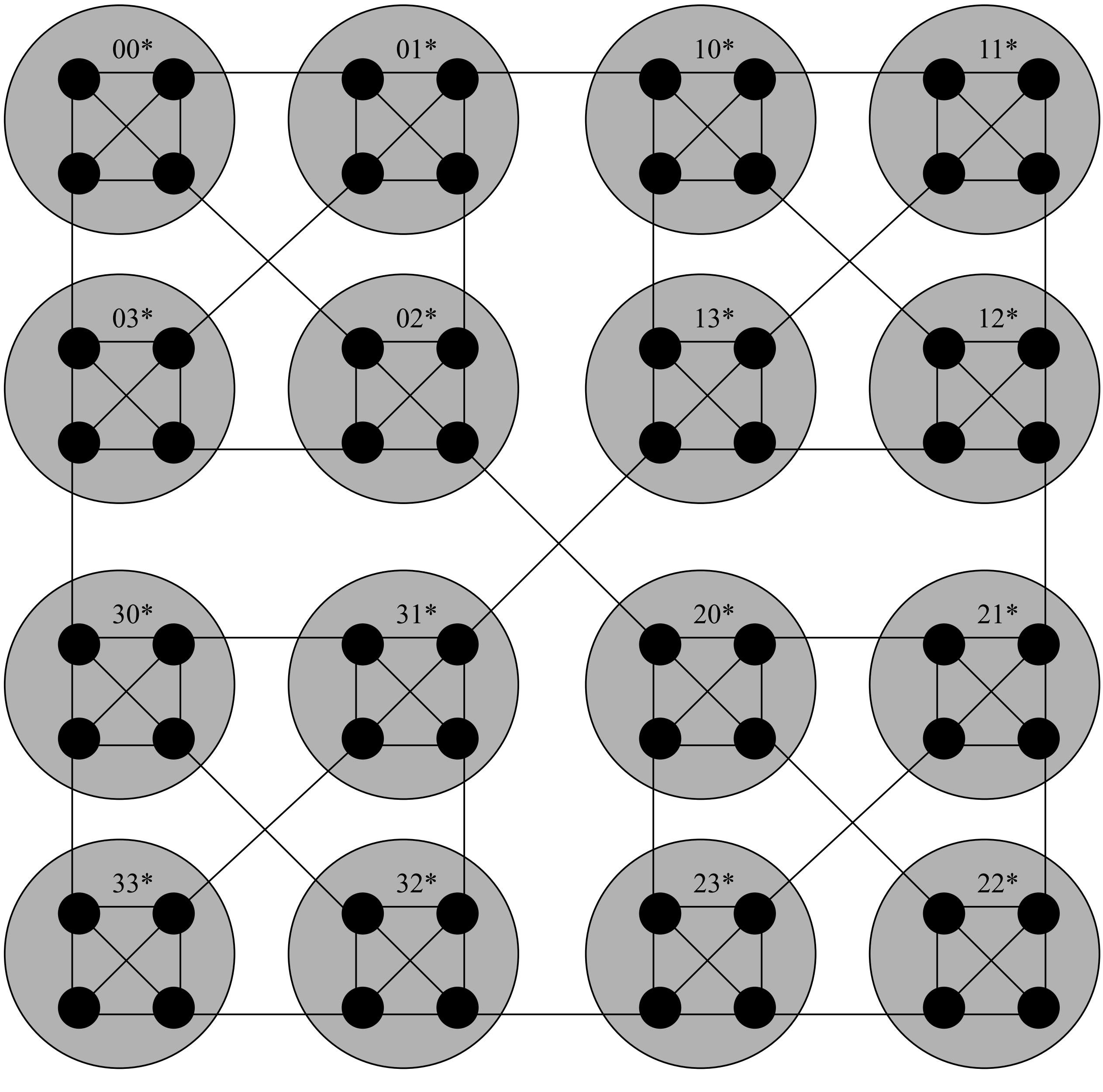


Figure 2. The outline graph of WK(4, 3).

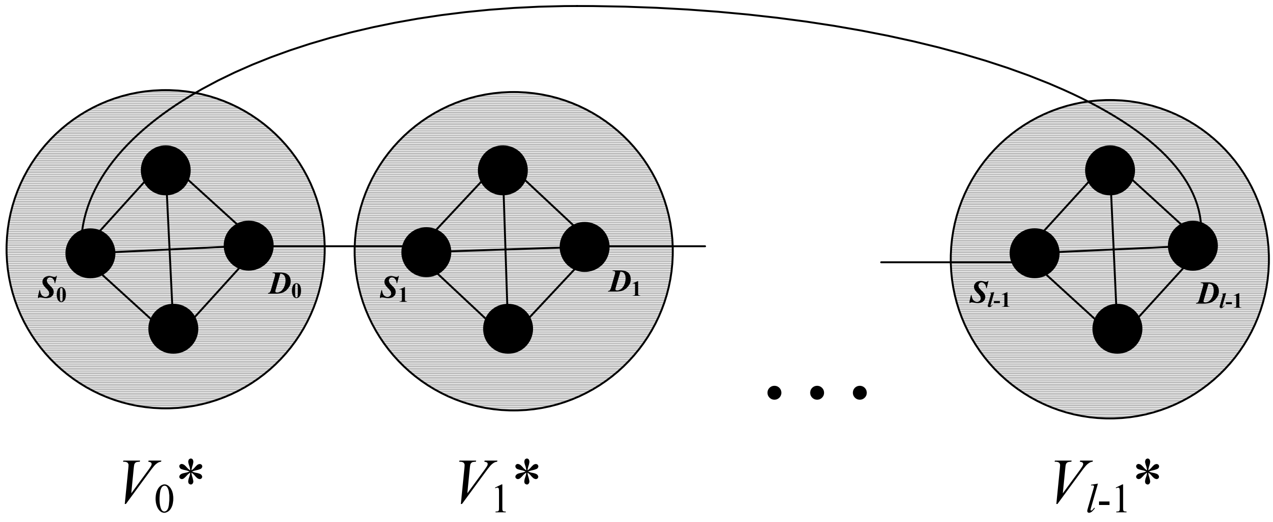


Figure 3. A cycle of length l in an OG(WK(W, L)).

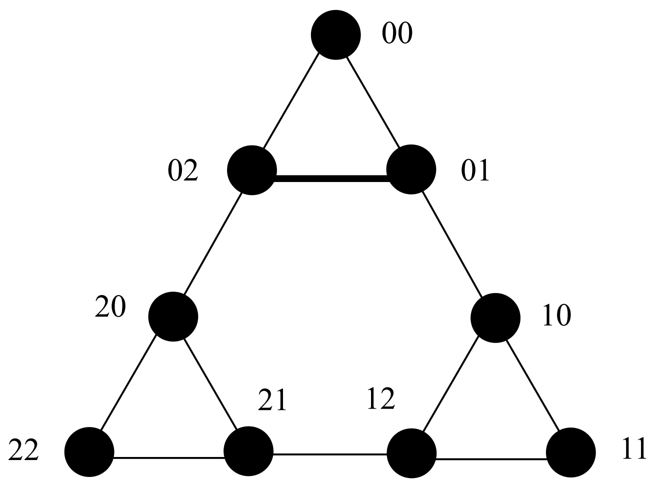


Figure 4. The edge (02, 01) cannot reside in a Hamiltonian cycle of a WK(3, 2).