Conditional Diagnosability of Cayley Graphs Generated by Transposition Trees under the Comparison Diagnosis Model[*]

Cheng-Kuan Lin
Department of Computer Science
National Chiao Tung University
cklin@cs.nctu.edu.tw

Jimmy J. M. Tan
Department of Computer Science
National Chiao Tung University
jmtan@cs.nctu.edu.tw

Lih-Hsing Hsu
Department of Computer Science and Information Engineering
Providence University
lhhsu@pu.edu.tw

Eddie Cheng
Department of Mathematics and Statistics
Oakland University
echeng@oakland.edu

László Lipták
Department of Mathematics and Statistics
Oakland University
liptak@oakland.edu

Abstract

The diagnosis of faulty processors plays an important role in multiprocessor systems for reliable computing, and the diagnosability of many well-known networks has been explored. Zheng et al. showed that the diagnosability of the n-dimensional star graph S_n is $n - 1$. Lai et al. introduced a restricted diagnosability of multiprocessor systems called conditional diagnosability. They consider the situation when no faulty set can contain all the neighbors of any vertex in the system. In this paper, we study the conditional diagnosability of Cayley graphs generated by transposition trees (which include the star graphs) under the comparison model, and show that it is $3n - 8$ for $n \geq 4$, except for the n-dimensional star graph, for which it is $3n - 7$.

1 Introduction

With the continuous increase in the size of multiprocessor systems, working in multiprocessor systems with faults has become unavoidable. Therefore, the problem of fault diagnosis in multiprocessor systems has gained increasing importance and has been widely studied, for example [9–11, 20, 21, 38, 39]. The process of identifying

1

faulty processors in a system is known as system-level diagnosis. Several different approaches have been developed to diagnose faulty processors, among which there are two fundamental approaches on system-level diagnosis. One major approach is called the comparison model, proposed by Malek and Maeng [28, 29]. In this model, each processor performs a diagnosis by sending the same inputs to each pair of its distinct neighbors and then compares their responses. The result of a comparison is either that the two responses agree or the two responses disagree. Based on the results of all the comparisons, one needs to decide the faulty or non-faulty (fault-free) status of the processors in the system. Another major approach is the PMC model established by Preparata, Metze, and Chien [33]. In this model, it is assumed that a processor can test the faulty or fault-free status of another adjacent processor. Under the PMC model, only processors with a direct link are allowed to test each other. It is assumed that if a processor is fault-free, it always gives correct and reliable testing results, and if a processor is faulty, then its testing results may be correct or incorrect. By analyzing the collection of all testing results, all of the faulty processors need to be identified.

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The n-cube is one of the most popular topologies [23, 35]. The n-dimensional star network S_n was proposed in [1] as "an attractive alternative to the n-cube" topology for interconnecting processors in parallel computers. Since its introduction, the network S_n has received considerable attention.

The star graphs are bipartite, vertex transitive, and edge transitive, and several classes of graphs can be embedded into them, e.g. grids [19], trees [3, 5, 13], and hypercubes [30]. Cycle embeddings and path embeddings are studied in [15–19, 24, 32]. The diameter and fault diameters of star graphs were computed in [1, 22, 34]. Some other interesting properties of star graphs are studied in [12, 14, 25–27].

Reviewing some previous papers (see [10, 11, 21, 38]), the n-dimensional hypercube Q_n, the n-dimensional crossed cube CQ_n, the n-dimensional twisted cube TQ_n, and the n-dimensional möbius cube MQ_n, all have diagnosability n under the comparison model. Zheng et al. [39] showed that the diagnosability of the n-dimensional star graph S_n is $n - 1$. In classical measures of system-level diagnosability for multiprocessor systems, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. As a consequence, the diagnosability of a system is limited by its minimum degree. Hence Lai et al. introduced a restricted diagnosability of multiprocessor systems called *conditional diagnosability* in [20]. Lai et al. considered this measure by requiring that for each processor v in a system, all the processors that are directly connected to v do not fail at the same time. Under this condition, the conditional diagnosability of the n-dimensional hypercube Q_n is $4n - 7$ under the PMC model [20].

In this paper, we study the conditional diagnosability of the star graph S_n and a class of graphs that arise as a generalization of the star graph. These graphs are Cayley graphs generated by transposition trees. We consider the comparison model and show that the conditional diagnosability of these graphs is $3n - 8$ for $n \geq 4$, except for the n-dimensional star graph, for which it is $3n - 7$.

Hence the conditional diagnosability of these graphs is about three times larger than their classical diagnosability. Section 2 provides preliminaries and previous results for diagnosing a system. In Section 3 we study the conditional diagnosability of Cayley graphs generated by transposition trees under the comparison model. Our conclusions are given in Section 4.

2 Preliminaries

A multiprocessor system can be represented by a *graph* $G(V, E)$, where the set of vertices $V(G)$ represents processors and the set of edges $E(G)$ represents communication links between processors. Throughout this paper, we focus on undirected graphs without loops and follow [4] for graph theoretical definitions and notations.

Let G be a graph. The *neighborhood* $N_G(v)$ of vertex v in G is the set of all vertices that are adjacent to v. The cardinality $|N_G(v)|$ is called the *degree* of v, denoted by $\deg_G(v)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a subset of $V(G) \cup E(G)$. The subgraph of G *induced* by S, denoted by $G[S]$, is the graph with the vertex set $S \cap V(G)$ and the edge set $\{(u, v) \mid (u, v) \in E(G) \text{ and } u, v \in S\}$. For a set of vertices (respectively, edges) S, we use the notation $G - S$ to denote the graph obtained from G by removing all the vertices (respectively, edges) in S. The components of G are its maximal connected subgraphs. A component is *trivial* if it has no edges; otherwise, it is nontrivial. The *connectivity* $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or a trivial graph. A graph G is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. A *path* P between vertices v_1 and v_k is a sequence of adjacent vertices, $\langle v_1, v_2, \ldots, v_k \rangle$, in which the vertices $v_1, v_2, \ldots, v_k$ are distinct. The *length* of P, denoted by $l(P)$, is the number of edges in P. The *dis-*

tance of two vertices u and v of G, denoted by $d_G(u, v)$, is the length of the shortest path of G between u and v.

The comparison diagnosis model [28,29] was proposed by Malek and Maeng. In this model, a self-diagnosable system is often represented by a multigraph $M(V, C)$, where V is the same vertex set defined in G, and C is a labeled edge set. If (u, v) is an edge labeled by w, then the labeled edge $(u, v)_w$ is said to belong to C, which implies that vertices u and v are being compared by vertex w. The same pair of vertices may be compared by different comparators, so M can be a multigraph. For $(u, v)_w \in C$, we use $r((u, v)_w)$ to denote the result of comparing vertices u and v by w such that $r((u, v)_w) = 0$ if the outputs of u and v agree, and $r((u, v)_w) = 1$ if the outputs disagree. In this model, if $r((u, v)_w) = 0$ and w is fault-free, then both u and v are fault-free. If $r((u, v)_w) = 1$, then at least one of the three vertices u, v, w must be faulty. If the comparator w is faulty, then the result of comparison is unreliable. The collection of all comparison results, given by the function $r : C \to \{0, 1\}$, is called the *syndrome* of the diagnosis. A subset $F \subset V$ is said to be *compatible* with a syndrome r if r can arise from the circumstance that all vertices in F are faulty and all vertices in $V - F$ are fault-free. A system is said to be *diagnosable* if, for every syndrome r, there is a unique $F \subset V$ that is compatible with r.

In our comparison model, we have $(u, v)_w \in C$ if and only if u and v are both adjacent to w, hence the original graph determines the multigraph $M(V, C)$. Notice that in this model for every set $F \subset V$ there is always a syndrome that is compatible for both F and $V - F$. Thus in general there is no diagnosable system. Thus [36] introduced the concept of a t-diagnosable system, in which the system is diagnosable as long as the number of faulty vertices does not exceed t. The max-

imum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$. A faulty comparator can lead to unreliable results, so a set of faulty vertices may produce different syndromes. Let $\sigma_F = \{\sigma \mid \sigma$ is compatible with $F\}$. Two distinct subsets F_1 and F_2 of V are said to be *indistinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} \neq \emptyset$; otherwise, F_1 and F_2 are said to be *distinguishable*. There are several different ways to verify whether a system is t-diagnosable under the comparison approach. The *symmetric difference* of the two sets S_1 and S_2 is defined as the set $S_1 \triangle S_2 = (S_1 - S_2) \cup (S_2 - S_1)$. The following theorem given by Sengupta and Dahbura [36] is a necessary and sufficient condition for ensuring distinguishability.
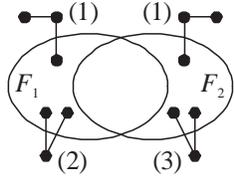
Figure 1: Description of distinguishability for Theorem 1

Theorem 1. *[36] Let G be a graph. For any two distinct subsets F_1 and F_2 of $V(G)$, (F_1, F_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied (see Figure 1):*

(1) *there are two distinct vertices u and w in $V(G) - (F_1 \cup F_2)$ and there is a vertex v in $F_1 \triangle F_2$ such that $(u, v)_w \in C$,*

(2) *there are two distinct vertices u and v in $F_1 - F_2$ and there is a vertex w in $V(G) - (F_1 \cup F_2)$ such that $(u, v)_w \in C$, or*

(3) *there are two distinct vertices u and v in $F_2 - F_1$ and there is a vertex w in $V(G) - (F_1 \cup F_2)$ such that $(u, v)_w \in C$.*

3 Transposition trees graphs

In this section we summarize the connectivity properties of Cayley graphs generated by transposition trees. These graphs arise naturally as a common generalization of star graphs and bubble-sort graphs. Some papers studying these graphs include [2, 6–8, 37].

Let Γ be a finite group and S be a set of elements of Γ such that the identity of the group does not belong to S. The *Cayley graph* $\Gamma(S)$ is the directed graph whose vertex set is Γ, and there is an arc from u to v if and only if there is an $s \in S$ such that $u = vs$. The graph $\Gamma(S)$ is connected if and only if S is a generating set for Γ. A Cayley graph is always vertex transitive, so it is maximally arc-connected if it is connected; however, its vertex connectivity may be low.

In this paper, we choose the finite group to be Γ_n, the symmetric group on $\{1, 2, \dots, n\}$, and the generating set S to be a set of transpositions. The vertices of the corresponding Cayley graph are permutations, and since S only has transpositions, there is an arc from vertex u to vertex v if and only if there is an arc from v to u. Hence we can regard these Cayley graphs as undirected graphs by replacing every pair of arcs between two vertices with an edge; let the resulting graph be $\Gamma_n(S)$. A simple way to depict S is via a graph $G(S)$ with vertex set $\{1, 2, \dots, n\}$, where there is an edge between i and j if and only if the transposition (ij) belongs to S. This graph is called the *transposition generating graph* of $\Gamma_n(S)$ or simply *transposition (generating) graph* if it is clear from the context. In fact, the star graph S_n was introduced via the generating graph $K_{1, n-1}$, where the center is 1 and the leaves

4

are $2, 3, \ldots, n$. Notice, that if we change the label of the center, we still get a graph isomorphic to the star graph S_n, hence with a slight abuse of terminology we will call all these graphs star graphs. The star graphs S_2, S_3, and S_4 are shown in Figure 2 for illustration.
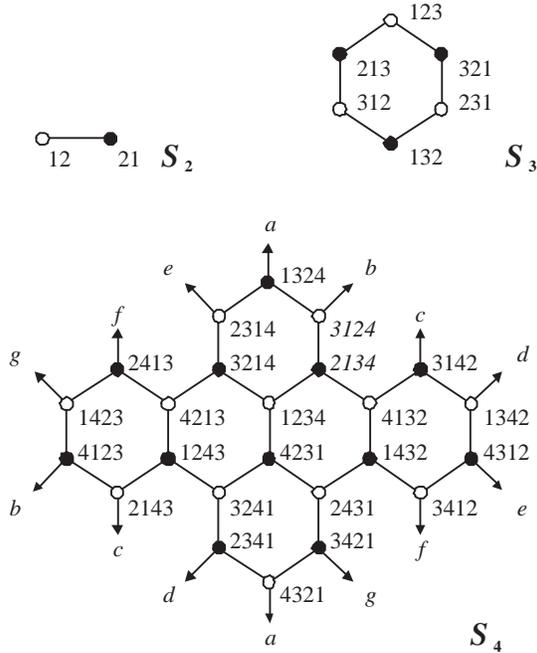
Figure 2: The star graphs S_2, S_3, and S_4

Note that the Cayley graph $\Gamma_n(S)$ is $|S|$-regular, and it is connected if and only if the generating graph $G(S)$ is connected. Since an interconnection network needs to be connected, we require the transposition graph to be connected. Here we will only consider the fundamental case, when $G(S)$ is a tree, and call the corresponding transposition generating graph a *transposition tree*. Thus the Cayley graphs obtained by these transposition trees are $(n-1)$-regular and have $n!$ vertices. In addition to the star graph mentioned above, these Cayley graphs also include

the bubble-sort graph whose transposition tree is a path. Figure 3 shows the bubble-sort graph for $n = 4$.
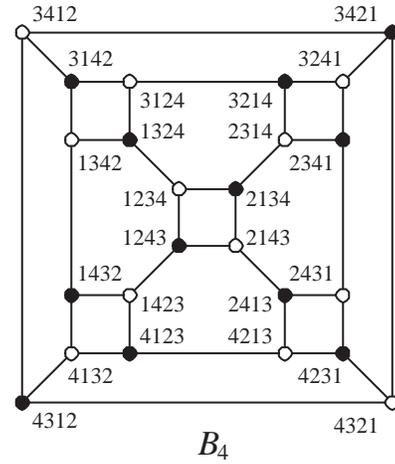
Figure 3: The bubble-sort graph

Let $\Gamma_n(S)$ be a Cayley graph generated by a transposition tree S. To help us describe the structure of the Cayley graph $\Gamma_n(S)$ when $G(S)$ is a tree, without loss of generality we may assume that a leaf of the transposition tree is n. We use boldface letters to denote vertices in $\Gamma_n(S)$. Hence, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a sequence of n vertices in $\Gamma_n(S)$. It is known that the connectivity of $\Gamma_n(S)$ is $n - 1$. Clearly $\Gamma_n(S)$ is a bipartite graph with one partite set containing the vertices corresponding to odd permutations and the other partite set containing the vertices corresponding to even permutations. Let $\mathbf{u} = u_1 u_2 \ldots u_n$ be any vertex of the Cayley graph $\Gamma_n(S)$. We say that u_i is the *i-th coordinate* of $\mathbf{u}$, denoted by $(\mathbf{u})_i$, for $1 \leq i \leq n$. For $1 \leq i \leq n$, let $\Gamma_n^{\{i\}}$ denote the subgraph of $\Gamma_n(S)$ induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$.

Since n is a leaf in the generating tree, it is easy to see that the Cayley graph $\Gamma_n(S)$ has the following properties:

(I) $\Gamma_n(S)$ consists of n vertex-disjoint subgraphs:

5

$\Gamma_n^{\{1\}}, \Gamma_n^{\{2\}}, \ldots, \Gamma_n^{\{n\}}$; each isomorphic to another Cayley graph $\Gamma_{n-1}(S')$ with $S' = S \setminus \{\pi\}$ where π is the transposition corresponding to the edge incident to the leaf n.

(II) $\Gamma_n^{\{i\}}$ has $(n-1)!$ vertices, and it is $(n-2)$-regular for all i.

(III) For all i, each vertex in $\Gamma_n^{\{i\}}$ has a unique neighbor outside $\Gamma_n^{\{i\}}$, and these outside neighbors are all different. There are exactly $(n-2)!$ independent edges between $\Gamma_n^{\{i\}}$ and $\Gamma_n^{\{j\}}$ for all $i \neq j$.

These properties are illustrated in Figures 2 and 3, as e.g. S_4 and the bubble-sort graph contain four copies of a smaller Cayley graph, the 6-cycle. Note that the 6-cycle is the shortest cycle in star graphs, whereas in other Cayley graphs we also have 4-cycles.

Cayley graphs generated by transposition trees have strong connectivity properties. Roughly speaking, deleting a large number of vertices from it, they will still contain a large connected component as shown by the following theorem:

Theorem 2. *[8] Let $\Gamma_n(S)$ be a Cayley graph obtained from a transposition generating tree S on $\{1, 2, \ldots, n\}$ with $n \geq 4$, and let T be a set of vertices of G such that $|T| \leq 3n - 8$. Then $\Gamma_n(S) - T$ satisfies one of the following conditions:*

(i) $\Gamma_n(S) - T$ *is connected.*

(ii) $\Gamma_n(S) - T$ *has two components, one of which is K_1 or K_2.*

(iii) $\Gamma_n(S) - T$ *has three components, two of which are singletons.*

(iv) $\Gamma_n(S) - T$ *has two components, one of which is a path of length 3, and T is the union of the neighbor sets of the vertices on the path except the vertices of the path itself with $|T| = 3n - 8$.*

(v) $\Gamma_n(S) - T$ *has four components, three of which are singletons, and T is the union of the neighbor sets of the singletons with $|T| = 3n - 8$.*

(vi) $\Gamma_n(S) - T$ *has two components, one of which is a 4-cycle, $n = 4$ and $|T| = 4$.*

Note: Cases (iv), (v), and (vi) can only occur when $\Gamma_n(S)$ is not a star graph, because each require a 4-cycle in the graph.

4 The conditional diagnosability

In classical measures of system-level diagnosability for multiprocessor systems, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. So the diagnosability of a system is limited by its minimum vertex degree. In particular, as we mentioned before, the star graph S_n has diagnosability $n - 1$ (see [39]). The same result can be proven easily for Cayley graphs generated by transposition trees as well, whose proof we omit:

Theorem 3. *Let $\Gamma_n(S)$ be a Cayley graph obtained from a transposition generating tree S on $\{1, 2, \ldots, n\}$ with $n \geq 4$. Then $t(\Gamma_n(S)) = n - 1$.*

A Cayley graph $\Gamma_n(S)$ has $\binom{n!}{n-1}$ vertex subsets of size $n-1$, among which there are only $n!$ vertex subsets which contain all the neighbors of some vertex. Since the ratio $n! / \binom{n!}{n-1}$ is very small for large n, in case of independent failures the probability of a faulty set containing all the neighbors of any vertex is very low. For this reason, Lai et al. introduced a new restricted diagnosability of multiprocessor systems called conditional diagnosability in [20]. They considered the situation that no faulty

set can contain all the neighbors of any vertex in a system. We need some terms to define the conditional diagnosability formally. A faulty set $F \subset V(G)$ is called a *conditional faulty set* if $N_G(v) \nsubseteq F$ for every vertex $v \in V(G)$. A system described by the graph $G(V, E)$ is said to be *conditionally t-diagnosable* if F_1 and F_2 are distinguishable for each pair of distinct conditional faulty sets F_1 and F_2 of $V(G)$ with $|F_1| \leq t$ and $|F_2| \leq t$. The maximum value of t such that G is conditionally t-diagnosable is called the *conditional diagnosability* of G, denoted by $t_c(G)$. It is trivial that $t_c(G) \geq t(G)$.

Now we give an example in the Cayley graph $\Gamma_n(S)$ to get a bound on the conditional diagnosability. As shown in Figure 4, we take a path of length two in
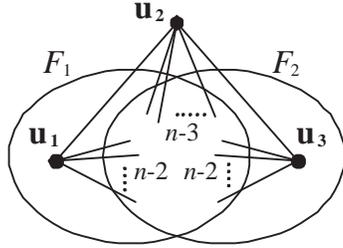
Figure 4: An indistinguishable conditional pair (F_1, F_2)

$\Gamma_n(S)$. Let $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$ be a path with length two. We set $A = N_{\Gamma_n(S)}(\mathbf{u}_1) \cup N_{\Gamma_n(S)}(\mathbf{u}_2) \cup N_{\Gamma_n(S)}(\mathbf{u}_3)$, $F_1 = A - \{\mathbf{u}_2, \mathbf{u}_3\}$ and $F_2 = A - \{\mathbf{u}_1, \mathbf{u}_2\}$. It is straightforward to check that F_1 and F_2 are two conditional faulty sets, and F_1 and F_2 are indistinguishable by Theorem 1. When $\Gamma_n(S)$ is a star graph, it has no cycles with length less than 6, hence the vertices in $N_{\Gamma_n(S)}(\mathbf{u}_1)$, $N_{\Gamma_n(S)}(\mathbf{u}_2)$, and $N_{\Gamma_n(S)}(\mathbf{u}_3)$ are all different, thus $|F_1| = |F_2| = 3n - 6$. On the other hand, if $\Gamma_n(S)$ is not a star graph, it contains 4-cycles, so some of those neighbors may be the same. However, it is easy to see that any two vertices in $\Gamma_n(S)$ can have at most two common neighbors. Thus when the path $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$ is part of a 4-cycle, we get $|F_1| = |F_2| = 3n - 7$. In both cases we have $|F_1 - F_2| = |F_2 - F_1| = 1$, therefore when $\Gamma_n(S)$ is a star graph, it is not conditionally $(3n - 6)$-diagnosable, otherwise $\Gamma_n(S)$ is not conditionally $(3n - 7)$-diagnosable. Hence we have the following result:

Proposition 4. *For $n \geq 4$, $t_c(\Gamma_n(S)) \leq 3n - 7$ when $\Gamma_n(S)$ is a star graph, otherwise $t_c(\Gamma_n(S)) \leq 3n - 8$.*

The following two lemmas will be needed to show our result on the conditional diagnosability of $\Gamma_n(S)$ for $n \geq 4$.

Lemma 5. *For $n \geq 4$, let F_1 and F_2 be any two distinct conditional faulty subsets of $V(\Gamma_n(S))$ with $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$ if $\Gamma_n(S)$ is a star graph, and $|F_1| \leq 3n - 8$ and $|F_2| \leq 3n - 8$ otherwise. Denote by H the maximum component of $\Gamma_n(S) - (F_1 \cap F_2)$. Then for every vertex $\mathbf{u}$ in $F_1 \triangle F_2$, $\mathbf{u}$ is in H.*

Proof. Without loss of generality, we assume that $\mathbf{u}$ is in $F_1 - F_2$. Since F_2 is a conditional faulty set, there is vertex $\mathbf{v}$ in $(V(\Gamma_n(S)) - F_2) - \{\mathbf{u}\}$ such that $(\mathbf{u}, \mathbf{v}) \in E(\Gamma_n(S))$. Suppose that $\mathbf{u}$ is not a vertex of H. Then $\mathbf{v}$ is not in $V(H)$, so $\mathbf{u}$ and $\mathbf{v}$ are part of a small component in $\Gamma_n(S) - (F_1 \cap F_2)$. Since F_1 and F_2 are distinct, we have $|F_1 \cap F_2| \leq 3n - 8$ when $\Gamma_n(S)$ is a star graph and $|F_1 \cap F_2| \leq 3n - 9$ otherwise. Thus in Theorem 2 cases (iv)–(vi) can't occur, hence $\{\mathbf{u}, \mathbf{v}\}$ forms a component K_2 of $\Gamma_n(S) - (F_1 \cap F_2)$, i.e. $\mathbf{u}$ is the unique neighbor of $\mathbf{v}$ in $\Gamma_n(S) - (F_1 \cap F_2)$. This is a contradiction since F_1 is a conditional faulty set, but all the neighbors of $\mathbf{v}$ are faulty in $\Gamma_n(S) - F_1$. $\square$

Lemma 6. *Let G be a graph with $\delta(G) \geq 2$, and let F_1 and F_2 be any two distinct conditional faulty subsets of $V(G)$ with $F_2 \subset F_1$. Then (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.*

Proof. Let u be any vertex of $F_1 - F_2$. Since F_1 is a conditional faulty subset of $V(G)$, there is a vertex v of $V(G) - F_1$ such that $(u, v) \in E(G)$ and there is a vertex w of $V(G) - F_1$ such that $(v, w) \in E(G)$. Since $F_2 \subset F_1$, neither v nor w is in F_2. By Theorem 1, (F_1, F_2) is a distinguishable pair. $\square$

Now we can prove our main results:

Theorem 7. *For $n \geq 4$, let F_1 and F_2 be two distinct conditional faulty subsets of $V(\Gamma_n(S))$. Assume that $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$ when $\Gamma_n(S)$ is a star graph, and $|F_1| \leq 3n - 8$ and $|F_2| \leq 3n - 8$ otherwise. Then (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.*

Proof. By Lemma 6, (F_1, F_2) is a distinguishable pair if $F_1 \subset F_2$ or $F_2 \subset F_1$. Thus we assume that $|F_1 - F_2| \geq 1$ and $|F_2 - F_1| \geq 1$. Let $A = F_1 \cap F_2$. Then we have $|A| \leq 3n - 8$ when $\Gamma_n(S)$ is a star graph, and $|A| \leq 3n - 9$ otherwise. Let H be the maximum component of $\Gamma_n(S) - A$. By Lemma 5, every vertex in $F_1 \triangle F_2$ is in H.

We claim that H has a vertex $\mathbf{v}$ outside $F_1 \cup F_2$ that has no neighbor in A. Since every vertex has degree $n - 1$, vertices in A can have at most $|A|(n-1)$ neighbors in H. There are at most $2(3n - 7) - |A|$ vertices in $F_1 \cup F_2$, and at most two vertices of $\Gamma_n(S) - A$ may not belong to H by Theorem 2. Since $|A| \leq 3n - 8$, we have $n! - |A|(n-2) - 2(3n - 7) - 2 \geq n! - (3n - 8)(n - 2) - 2(3n - 7) - 2 \geq 4$ when $n \geq 4$. Thus there must be vertices of H outside $F_1 \cup F_2$ having no neighbor in A; let $\mathbf{v}$ be such a vertex.

If $\mathbf{v}$ has no neighbor in $F_1 \cup F_2$, then we can find a path of length at least 2 within H to a vertex $\mathbf{p}$ in $F_1 \triangle F_2$. We may assume that $\mathbf{p}$ is the first vertex of $F_1 \triangle F_2$ on this path, and let $\mathbf{q}$ and $\mathbf{w}$ be the two vertices on this path immediately before $\mathbf{p}$ (we may have $\mathbf{v} = \mathbf{q}$), so $\mathbf{q}$ and $\mathbf{w}$ are not in $F_1 \cup F_2$. Then the edges $(\mathbf{q}, \mathbf{w})$ and $(\mathbf{w}, \mathbf{p})$ show that (F_1, F_2) is a distinguishable conditional pair. Now assume that $\mathbf{v}$ has a neighbor in $F_1 \triangle F_2$. Then since the degree of $\mathbf{v}$ is at least 3, and $\mathbf{v}$ has no neighbor in A, there are three possibilities:

(1) $\mathbf{v}$ has two neighbors in $F_1 - F_2$,

(2) $\mathbf{v}$ has two neighbors in $F_2 - F_1$, or

(3) $\mathbf{v}$ has at least one neighbor outside $F_1 \cup F_2$.

In each case Theorem 1 implies that (F_1, F_2) is a distinguishable conditional pair of $\Gamma_n(S)$ under the comparison diagnosis model, finishing the proof. $\square$

To summarize, with Proposition 4 and Theorem 7, we have the following result.

Theorem 8. *For $n \geq 4$, $t_c(\Gamma_n(S)) = 3n - 7$ when $\Gamma_n(S)$ is a star graph, and $t_c(\Gamma_n(S)) = 3n - 8$ otherwise.*

Remark: Theorem 3 can be proved similarly, indeed much simpler, using that its connectivity is $n - 1$, proved in [6].

5 Conclusions

In the real world, processors fail independently and with different probabilities. The probability that any faulty set contains all the neighbors of some processor is very small [31], so we are interested in the study of conditional diagnosability. A new diagnosis measure proposed by Lai et al. [20] requires that each processor of

8

a system is incident with at least one fault-free processor. In this paper, we considered Cayley graphs generated by transposition trees, which are a generalization of the n-dimensional star graph S_n, and showed that the conditional diagnosability of $\Gamma_n(S)$ is $3n - 8$ under the comparison model except when it is the star graph, for which the conditional diagnosability is $3n - 7$. This number is about three times as large as the classical diagnosability. It would be interesting to find other conditional measures for network reliability under which diagnosability of such networks are even higher.

References

[1] S. B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (1989) 555–566.

[2] T. Araki, Hyper hamiltonian laceability of cayley graphs generated by transpositions, Networks 48 (2006) 121–124.

[3] N. Bagherzadeh, M. Dowd, and N. Nassif, Embedding an arbitrary binary tree into the star graph, IEEE Transactions on Computers 45 (1996) 475–481.

[4] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North Holland, New York, 1980.

[5] A. Bouabdallah, M. C. Heydemann, J. Opatrny, and D. Sotteau, Embedding complete binary trees into star and pancake graphs, Theory of Computing Systems 31 (1998) 279–305.

[6] E. Cheng and L. Lipták, Fault resiliency of cayley graphs generated by transpositions, International Journal of Foundations of Computer Science, to appear.

[7] E. Cheng and L. Lipták, Linearly many faults in cayley graphs generated by transposition trees, Information Sciences, to appear.

[8] E. Cheng and L. Lipták, Structural properties of cayley graphs generated by transposition trees, Congressus Numerantium 180 (2006) 81–96.

[9] A. T. Dahbura and G. M. Masson, An $O(n^{2.5})$ faulty identification algorithm for diagnosable systems, IEEE Transactions on Computers 33 (1984) 486–492.

[10] J. Fan, Diagnosability of crossed cubes under the comparison diagnosis model, IEEE Transactions on Parallel and Distributed Systems 13 (2002) 1099–1104.

[11] J. Fan, Diagnosability of the möbious cubes, IEEE Transactions on Parallel and Distributed Systems 9 (1998) 923–928.

[12] P. Fragopoulou and S. G. Akl, Optimal communication algorithms on the star graphs using spanning tree constructions, Journal of Parallel and Distributed Computing 24 (1995) 55–71.

[13] P. Fragopoulou and S. G. Akl, Edge-disjoint spanning trees on the star networks with applications to fault tolerance, IEEE Transactions on Computers 45 (1996) 174–185.

[14] Q. P. Gu and S. Peng, Node-to-node cluster fault tolerant routing in star graphs, Information Processing Letters 56 (1995) 29–35.

[15] S.-Y. Hsieh, Embedding longest fault-free paths onto star graphs with more vertex faults, Theoretical Computer Science 337 (2005) 370–378.

[16] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, Hamiltonian-laceability of star graphs, Networks 36 (2000) 225–232.

[17] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, Longest fault-free paths in star graphs with vertex faults, Theoretical Computer Science 262 (2001) 215–227.

[18] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, Longest fault-free paths in star graphs with edge faults, IEEE Transactions on Computers 50 (2001) 960–971.

[19] J. S. Jwo, S. Lakshmivarahan, and S. K. Dhall, Embedding of cycles and grids in star graphs, Journal of Circuits, Systems, and Computers 1 (1991) 43–74.

[20] P.-L. Lai, J. J. M. Tan, C.-P. Chang, and L.-H. Hsu, Conditional diagnosability measures for large multiprocessor systems, IEEE Transactions on Computers 54 (2005) 165–175.

[21] P.-L. Lai, J. J. M. Tan, C-H. Tsai, and L.-H. Hsu, The diagnosability of the matching composition network under the comparison diagnosis model, IEEE Transactions on Computers 53 (2004) 1064–1069.

[22] S. Latifi, On the fault-diameter of the star graph, Information Processing Letters 46 (1993) 143–150.

[23] F. T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays · Trees · Hypercubes, Morgan Kaufmann, CA, San Mateo, 1992.

[24] T.-K. Li, J. J. M. Tan, and L.-H. Hsu, Hyper hamiltonian laceability on edge fault star graph, Information Sciences 165 (2004) 59–71.

[25] C.-K. Lin, H.-M. Huang, and L.-H. Hsu, The super connectivity of the pancake graphs and the super laceability of the star graphs, Theoretical Computer Science 339 (2005) 257–271.

[26] C.-K. Lin, H.-M. Huang, L.-H. Hsu, and S. Bau, Mutually independent hamiltonian paths in star networks, Networks 46 (2005) 100–117.

[27] C.-K. Lin, H.-M. Huang, D. F. Hsu, and L.-H. Hsu, On the spanning w-wide diameter of the star graph, Networks 48 (2006) 235–249.

[28] J. Maeng and M. Malek, A comparison connection assignment for self-diagnosis of multiprocessors systems, Proceedings of the 11th International Symposium on Fault Tolerant Computing (1981) 173–175, 1981.

[29] M. Malek, A comparison connection assignment for diagnosis of multiprocessors systems, Proc. 7th International Symposium Computer Architecture, (1980) 31–35.

[30] Z. Miller, D. Pritikin, and I. H. Sudborough, Near embeddings of hypercubes into Cayley graphs on the symmetric group, IEEE Transactions on Computers 43 (1994) 13–22.

[31] W. Najjar and J. L. Gaudiot, Network resilience: a measure of network fault tolerance, IEEE Transactions on Computers 39 (1990) 174–181 1990

[32] J.-H. Park and H.-C. Kim, Longest paths and cycles in gaulty star graphs, Journal of Parallel and Distributed Computing 64 (2004) 1286–1296.

[33] F. P. Preparata, G. Metze, and R. T. Chien, On the connection assignment problem of diagnosis sys-

tems, IEEE Transactions on Electronic Computers 16 (1967) 848–854.

[34] Y. Rouskov, S. Latifi, and P. K. Srimani, Conditional fault diameter of star graph networks, Journal of Parallel and Distributed Computing 33 (1996) 91–97.

[35] Y. Saad and M. H. Schultz, Topological properties of hypercubes, IEEE Transactions on Computers 37 (1998) 867–872.

[36] A. Sengupta and A. Dahbura, On self-diagnosable multiprocessor systems: diagnosis by the comparison approach, IEEE Transactions on Computers 41 (1992) 1386–1396.

[37] M. Tchuente, Generation of permutations by graphical exchanges, Ars Combinatoria 14 (1982) 115–122.

[38] D. Wang, Diagnosability of hypercubes and enhanced hypercubes under the comparison diagnosis model, IEEE Transactions on Computers 48 (1999) 1369–1374.

[39] J. Zheng, S. Latifi, E. Regentova, K. Luo, and X. Wu, Diagnosability of star graphs under the comparison diagnosis model, Information Processing Letters 93 (2005) 29–36.