On the Two-Equal-Disjoint Path Cover Problem of Twisted Cubes

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Abstract

Embedding of paths have attracted much attention in the parallel processing. Many-to-many communication is one of the most central issues in various interconnection networks. A graph G is globally two-equal-disjoint path coverable if for any two distinct pairs of vertices (u, v) and (x, y) of G, there exist two disjoint paths P_{uv} and P_{xy} satisfied that (1) P_{uv} joins u to v and P_{xy} joins x to y, (2) $|P_{uv}| = |P_{xy}|$, and (3) $V(P_{xy} \bigcup P_{xy}) = V(G)$. In this paper, we prove that TQ_n is globally 2-equal-disjoint path coverable for $n \geq 5$.

Keywords: Interconnection network; Twisted cube; disjoint path; k-equal-disjoint path cover, 2-equal-disjoint path coverable.

1 Introduction

For the graph definition and notation we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. A *path* of length k from x to y is a finite sequence of distinct vertices $\langle v_0, v_1, v_2, \ldots, v_k \rangle$, where $x = v_0$, $y = v_k$, and $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq k$. For convenience, we use the sequence $\langle v_0, \ldots, v_i, P, v_j, \ldots, v_k \rangle$, where $P = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ to denote the path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$. Note that it is possible that the path P has length 0. We can also write the path $\langle v_0, v_1, v_2, \cdots, v_k \rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \cdots, v_j, P_2, v_t, \cdots, v_k \rangle$, where P_1 is the path $\langle v_0, v_1, \cdots, v_i \rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \cdots, v_t \rangle$. We use $d(u, v)$ to denote the distance between u and v, i.e., the length of the shortest path joining u and v.

A path is a *Hamiltonian path* if it contains all vertices of G. A graph G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two distinct vertices. A *cycle* is a path (except the first vertex is the same as the last vertex) containing at least three vertices. A cycle of G is a *Hamiltonian cycle* if it contains all vertices. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Finding node-disjoint paths is one of the important issues of routing among nodes in various interconnection networks. Node-disjoint (abbreviated as disjoint) paths can be used to avoid communication congestion and provide parallel paths for an efficient data routing among nodes. Moreover, multiple disjoint paths can be more fault-tolerant of node or link failures and greatly enhance the transmission reliability. Disjoint paths generally fall into three categories: one-to-one, one-to-many, and many-to-many. The one-to-one disjoint path is built with one source and one destination. The one-to-many disjoint paths like a tree structure, they contain one source and many distinct destination nodes. The many-to-many disjoint paths involve $k, k \geq 1$, disjoint paths with k pairs distinct source and destination nodes.

A *disjoint path cover* in a graph G is to find disjoint paths containing all the vertices in G. For an embedding of linear arrays in a network, the cover implies every node can be participated in a pipeline computation. One-to-one disjoint path covers in recursive circulants [8] and one-to-many disjoint path covers in some hypercube-like interconnection networks [9] were studied. The many-to-many k-disjoint path cover is proposed by Park etc. in [10]. In this paper, we call such many-to-many k-disjoint path cover (abbreviated as k-disjoint path cover) as many-to-many k-equal-disjoint path cover (abbreviated as k-equal-disjoint path cover) that k disjoint paths have same lengths. The k disjoint paths with equal length implies that the parallel processing of k pipeline is guaranteed accurately. Furthermore, a graph is called globally k-equal-

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disjoint path coverable if there exists a k-equal-disjoint path cover for any k distinct source-destination pairs.

An n-dimensional Twisted cube, TQ_n[3], is a variation of hypercube. It has 2^n vertices and each vertex has the same degree n. The difference is that it diverts some edges which are called *twisted edge* and these edge reduce the distance between vertices.

Though some topological properties of Twisted cubes have been studied in the literature [1, 4, 5, 7, 11, 12, 13]. In this paper, we prove that the Twisted cube is globally two-equal-disjoint path coverable. In next section, we give the definition of two-equal-disjoint path coverable problem and Twisted Cubes. Then we prove that the Twisted cube is globally two-equal-disjoint path coverable in the section 3. In the final section, we give the conclusion.

2 Preliminary

In this section, we will first give the definition of globally two-equal-disjoint path coverable problem of a graph G, and then we will give the relevant definitions in graph theory and the definition of the Twisted cubes.

Definition 1 *A graph G is (u, v, x, y)-two-equal-disjoint path coverable if there are two disjoint paths P_{uv} and P_{xy} such that P_{uv} joins the vertices u to v, P_{xy} joins the vertices x to y, and $V(P_{uv} \bigcup P_{xy}) = V(G)$.*

Definition 2 *A graph G is globally two-equal-disjoint path coverable if for any two distinct pairs of vertices (u, v) and (x, y), the (u, v, x, y)-two-equal-disjoint path cover exists.*

To define the Twisted cubes, as proposed by Hilbers [3], the term so called ”parity” is introduced.

Definition 3 *The parity $P_j(u) = u_j \bigoplus u_{j-1} \bigoplus \cdots \bigoplus u_1 \bigoplus u_0$ is a function using exclusive-or, $\bigoplus$.*

A *2-bit* in u is a pair of adjacent bits where the larger index is even, for example u_2u_1. And the i-th 2-bit is $u_{2i}u_{2i-1}$ for $i \geq 1$. But the 0-th 2-bit is just u_0. The following is the recursive definition of the n-dimensional Twisted cube TQ_n.

Definition 4 [3] *An n-dimensional twisted cube for $n = 2k + 1$, denoted TQ_n, is a graph $G(V, E)$, where $V = \{v | v \in \{0, 1\}^n\}$ and $E = \{(x, y) | x, y \in V$, satisfying*

(1) $x_{2i}x_{2i-1} = \overline{y}_{2i}y_{2i-1}$ *or* $y_{2i}\overline{y}_{2i-1}$*; with* $P_{2i-2}(x) = 1$*, or*
$x_{2i}x_{2i-1} = \overline{y}_{2i}y_{2i-1}$ *or* $\overline{y}_{2i}\overline{y}_{2i-1}$*; with* $P_{2i-2}(x) = 0$ *for some 2-bit i; and*

(2) $x_{2j}x_{2j-1} = y_{2j}y_{2i-1}$ *for any other 2-bits $j \neq i$.*

A *matching* M of a graph G is a set of pairwise disjoint edges. M is a perfect matching if each vertex of G belongs to some edge in M. From the definition, we have the following lemma.

Lemma 1 [6]
Both the subgraph induced by $TQ^{0,0}_{n-2} \bigcup TQ^{1,0}_{n-2}$ and the subgraph induced by $TQ^{0,1}_{n-2} \bigcup TQ^{1,1}_{n-2}$ are isomorphic to $TQ_{n-2} \times K_2$ where K_2 is the complete graph with two vertices. Moreover, the edges joining $TQ^{0,0}_{n-2} \bigcup TQ^{1,0}_{n-2}$ and $TQ^{0,1}_{n-2} \bigcup TQ^{1,1}_{n-2}$ form a perfect matching of TQ_n .

Let G and H be two graphs having the same number of vertices. $G \oplus_M H$ denotes a graph which has copies of G and H connected by a matching M. Let TQ^0_{n+1} and TQ^1_{n+1} be the subgraphs induced by $V(TQ^{0,0}_n) \bigcup V(TQ^{1,0}_n)$ and $V(TQ^{0,1}_n) \bigcup V(TQ^{1,1}_n)$, respectively. Then by Lemma 1, both of TQ^0_{n+1} and TQ^1_{n+1} are isomorphic to $TQ_n \times K_2$, and $TQ^0_{n+1} \oplus_M TQ^1_{n+1}$ is isomorphic to TQ_{n+2} for a specific matching M. In addition, $TQ_n \times K_2$ has two copies of TQ_n, and we use TQ^0_n and TQ^1_n to denote them, respectively. For convenience of discussion, we add 0 to every vertex $v \in V(TQ^0_n)$ and 1 to every vertex $u \in V(TQ^0_n)$, respectively, as the leading bits. As a result, each vertex of $TQ_n \times K_2$ is represented by a binary string of length $n + 1$. These notations are used extensively throughout this paper.

We then introduce two important fault Hamiltonian results for proving the main theorem in the next section of this paper. A graph G is k-fault Hamiltonian connected if for any faulty set $F \subset V(G) \cup E(G)$ such that $|F| \leq k$, $G - F$ is still Hamiltonian connected.

Lemma 2 [6] *TQ_n is $n-3$ fault Hamiltonian connected.*

Lemma 3 [6] *For $n \geq 3$ and $i \in \{0, 1\}$, $TQ^{0,i}_n \bigcup TQ^{1,i}_n$ is $(n-1)$-Hamiltonian and $(n-2)$-Hamiltonian connected*

3 Twisted cube is globally two-disjoint equal path coverable

As a starting point we present lemmas 4 and 5 which establish the base case of Theorem 1.

Lemma 4 *TQ_3 is not globally two-equal-disjoint path coverable.*

Proof. To prove this lemma, we give a counterexample. Given two pair of vertices $0, 1$ and $2, 4$, TQ_3 is not $(0, 1, 2, 4)$-two-equal-disjoint path coverable. □

Lemma 5 *TQ_5 is globally two-equal-disjoint path coverable.*

Proof. To prove this case is very tedious. With long and detail discussion, we have completed theoretical proof for TQ_5. Nevertheless, we do not present it in this paper for reducing complexity. However, we can also verify this small case directly using computer. □

Next we show $TQ_n \times K_2$ has the same result if TQ_n is globally two-equal-disjoint path coverable.

Lemma 6 *Let $n \geq 5$. $TQ_n \times K_2$ is also globally two-equal-disjoint path coverable if TQ_n is globally two-equal-disjoint path coverable.*

Proof. Let $TQ_{n+1} = TQ_n \times K_2$ and let TQ_n^0 and TQ_n^1 be the two components of TQ_{n+1} . Let (u, v) and (x, y) be any two distinct vertex pairs of TQ_{n+1}. Herein, we want to establish two disjoint paths P_{uv}, and P_{xy} with equal length $2^n - 1$. The proof consists of four cases as follows:

Case 1: All four end vertices belong to the same TQ_n^i, $i = 0, 1$.

Without loss of generality, let $u, v, x, y \in V(TQ_n^0)$. By hypothesis, there exist two equal disjoint paths P_{uv}^0 and P_{xy}^0 of length $2^{n-1} - 1$ in TQ_n^0. Let t (resp. w) be the neighbor of u (resp. x) on path P_{uv}^0 (resp. P_{xy}^0). Let $P_{uv}^0 = (u, t, P_{tv}^0)$ and $P_{xy}^0 = (x, w, P_{wy}^0)$. Let u^1, t^1, x^1, and w^1 be the neighbors of u, t, x, and w in TQ_n^1, respectively. Then, there are also two equal disjoint paths $P_{u^1t^1}^1$ and $P_{x^1w^1}^1$ of length $2^{n-1} - 1$ in TQ_n^1. Let $P_{uv} = (u, u^1, P_{u^1t^1}^1, t^1, t, P_{tv}^0, v)$ and $P_{xy} = (x, x^1, P_{x^1w^1}^1, w^1, w, P_{wy}^0, y)$. Hence, P_{uv} and P_{xy} are two equal disjoint paths of length $2^n - 1$ in $TQ_n \times K_2$.

Case 2: Three end vertices belong to TQ_n^i and another end vertex belongs to TQ_n^{1-i}, $i = 0, 1$.

Without loss of generality, let $u, v, x \in V(TQ_n^0)$ and $y \in V(TQ_n^1)$. Let y^0 be the neighbors of y in TQ_n^0. Let w be a vertex in TQ_n^0 and $w \notin \{u, v, x, y^0\}$. By hypothesis, there exist two equal disjoint paths P_{uv}^0 and P_{xw}^0 of length $2^{n-1} - 1$ in TQ_n^0. Let $(s, t) \in P_{uv}^0$ and $y^0 \notin \{s, t\}$. Let $P_{uv}^0 = \langle u, P_{us}^0, s, t, P_{tv}^0, v\rangle$.

Let s^1, t^1, w^1 be the neighbors of s, t, w in TQ_n^1, respectively. Note that there exist two equal disjoint paths $P_{s^1t^1}^1$ and $P_{w^1y}^0$ of length $2^{n-1} - 1$ in TQ_n^1. Let $P_{uv} = \langle u, P_{us}^0, s, s^1, P_{s^1t^1}^1, t^1, t, P_{tv}^0, v\rangle$ and $P_{xy} = \langle x, P_{xw}, w, w^1, P_{w^1y}, y\rangle$. Then, P_{uv} and P_{xy} are two equal disjoint paths of length $2^n - 1$ in $TQ_n \times K_2$.

Case 3: u, v belong to TQ_n^i and x, y belong to TQ_n^{1-i}, $i = 0, 1$.

Without loss of generality, let $u, v \in V(TQ_n^0)$ and $x, y \in V(TQ_n^1)$. By Lemma 2, there exist two paths P_{uv} and P_{xy} of equal length $2^n - 1$ in TQ_n^0 and TQ_n^1, respectively.

Case 4: u, x (or u, y) belong to TQ_n^i and v, y (or v, x) belong to TQ_n^{1-i}, $i = 0, 1$.

Without loss of generality, let $u, x \in V(TQ_n^0)$ and $v, y \in V(TQ_n^1)$. Let v^0, y^0 be the neighbors of v, y in TQ_n^0, respectively. Let s, w be two vertices except u, x, v^0, y^0 in TQ_n^0 and let s^1, w^1 be the neighbors of s, w in TQ_n^1, respectively. Let P_{us} and P_{xw} be two disjoint paths of length $2^{n-1} - 1$ in TQ_n^0. Let P_{s^1v} and P_{w^1y} be two disjoint paths of length $2^{n-1} - 1$ in TQ_n^1. Let $P_{uv} = \langle u, P_{us}, s, s^1, P_{s^1v}, v\rangle$ and $P_{xy} = \langle x, P_{xw}, w, w^1, P_{w^1y}, y\rangle$. Then P_{uv} and P_{xy} be two disjoint paths of lengths $2^n - 1$ in $TQ_n \times K_2$.

□

Next we formally show the main result that TQ_n, $n \geq 5$, is globally two-disjoint equal path coverable.

Theorem 1 *Twisted cube, TQ_n, is globally two-equal-disjoint path coverable for $n \geq 5$.*

Proof. We prove this theorem by induction on n. The base case is TQ_5. With Lemma 5, the base case holds. By induction hypothesis, we can assume that TQ_n is globally two-equal-disjoint path coverable. Now, we need to show that TQ_{n+2} is also globally two-equal-disjoint path coverable. Let u, v and x, y be two distinct source-destination pairs of TQ_{n+2}. In the following, we establish two disjoint paths P_{uv}, P_{xy} of length $2^{n+1} - 1$. Herein, we divide the proof into two cases according to which subgraphs, $TQ_n^{0,i} \bigcup TQ_n^{1,i}$, the four vertices exactly belong to as follows.

Case 1: The four vertices u, v, x, y exactly belong to one of the two subgraphs $TQ_n^{0,i} \bigcup TQ_n^{1,i}$ and $TQ_n^{1,1-i} \bigcup TQ_n^{1,1-i}$.

Without loss of generality, let $i = 0$. By Lemma 6, there exist two disjoint paths, denoted as P_{uv}^0 and P_{xy}^0, of length $2^n - 1$ in $TQ_n^{0,0} \bigcup TQ_n^{1,0}$. Let s and w be the neighbor of u and x on paths P_{uv}^0 and P_{xy}^0, respectively. Let $P_{uv}^0 = \langle u, s, P_{sv}^0, v\rangle$ and $P_{xy}^0 = \langle x, w, P_{wy}^0, y\rangle$. Let u^1, s^1, x^1, w^1 be the neighbors of vertices u, s, x, w in $TQ_n^{0,1} \bigcup TQ_n^{1,1}$, respectively. By Lemma 6, there also exist two disjoint paths, denoted as $P_{u^1s^1}^1$ and $P_{x^1w^1}^1$, of length $2^n - 1$ in $TQ_n^{0,1} \bigcup TQ_n^{1,1}$. Hence, $P_{uv} = \langle u, u^1, P_{u^1s^1}^1, s^1, s, P_{sv}^0, v\rangle$ and $P_{xy} = \langle x, x^1, P_{x^1w^1}^1, w^1, w, P_{wy}^0, y\rangle$ are two disjoint paths of length $2^{n+1} - 1$ in TQ_{n+2}.

Case 2: The four vertices u, v, x, y exactly belong to the two subgraphs $TQ_n^{0,0} \bigcup TQ_n^{1,0}$ and $TQ_n^{0,1} \bigcup TQ_n^{1,1}$.

Herein, we in advanced divide this case into three subcases as follows.

Subcase 2.1: Three end vertices belong to $TQ_n^{0,i} \bigcup TQ_n^{1,i}$ and another vertex belongs to $TQ_n^{0,1-i} \bigcup TQ_n^{1,1-i}$.

Without loss of generality, let $u, v, x \in V(TQ_n^{0,0}) \bigcup V(TQ_n^{1,0})$ and $y \in V(TQ_n^{0,1}) \bigcup V(TQ_n^{1,1})$. Let vertex $y^0 \in V(TQ_n^{0,0}) \bigcup V(TQ_n^{1,0})$ be a neighbor of y. Let w be a neighbor of x in $TQ_n^{0,0} \bigcup TQ_n^{1,0}$ except u, v and y^0. By Lemma 3, there exist path P_{uv}^0 of length $2^{n+1} - 3$ in $TQ_n^{0,0} \bigcup TQ_n^{1,0}$ except w and x. Let (s, t) be a edge on path P_{uv}^0 with $y^0 \notin \{s, t\}$. Let $P_{uv}^0 = \langle u, P_{us}^0, s, t, P_{tv}^0, v \rangle$. Let vertices $s^1, t^1, w^1 \in V(TQ_n^{0,1}) \bigcup V(TQ_n^{1,1})$ be the neighbors of s, t, w, respectively. By Lemma 3, there exist path P_{w^1y} of length $2^{n+1} - 3$ in $TQ_n^{0,1} \bigcup TQ_n^{1,1}$ except s^1 and t^1.

Let $P_{uv} = \langle u, P_{us}^0, s, s^1, t^1, t, P_{tv}^0, v \rangle$ and $P_{xy} = \langle x, w, w^1, P_{w^1y}, y \rangle$. Clearly, P_{uv} and P_{xy} are two disjoint paths of length $2^{n+1} - 1$.

Subcase 2.2: Each pair has one vertex in $TQ_n^{0,i} \bigcup TQ_n^{1,i}$ and another vertex in $TQ_n^{0,1-i} \bigcup TQ_n^{1,1-i}$.

Without loss of generality, let $u, x \in V(TQ_n^{0,0}) \bigcup V(TQ_n^{1,0})$ and $v, y \in V(TQ_n^{0,1}) \bigcup V(TQ_n^{1,1})$. Let $u^1, x^1 \in V(TQ_n^{0,1}) \bigcup V(TQ_n^{1,1})$ and $v^0, y^0 \in V(TQ_n^{0,0}) \bigcup V(TQ_n^{1,0})$ be the neighbors of u, v and v, y, respectively. Let s and w be two vertices in $TQ_n^{0,0} \bigcup TQ_n^{1,0}$ except u, x, v^0, y^0. Let s^1 and w^1 be the neighbors of s and w, respectively, in $TQ_n^{0,1} \bigcup TQ_n^{1,1}$.

By Lemma 6, there exist two disjoint paths P_{us}^0, P_{xw}^0 in $TQ_n^{0,0} \bigcup TQ_n^{1,0}$ and exist two disjoint paths $P_{s^1v}^1$, $P_{w^1y}^1$ in $TQ_n^{0,1} \bigcup TQ_n^{1,1}$ of length $2^n - 1$, respectively. Then $P_{uv} = \langle u, P_{us}^0, s, s^1, P_{s^1v}^1, v \rangle$ and $P_{xy} = \langle x, P_{xw}^0, w, w^1, P_{w^1y}^1, y \rangle$ are two disjoint paths of length $2^{n+1} - 1$.

Subcase 2.3: One pair belongs to $TQ_n^{0,i} \bigcup TQ_n^{1,i}$ and another pair belongs to $TQ_n^{0,1-i} \bigcup TQ_n^{1,1-i}$.

Without loss of generality, let $u, v \in V(TQ_n^{0,0}) \bigcup V(TQ_n^{1,0})$ and $x, y \in V(TQ_n^{0,1}) \bigcup V(TQ_n^{1,1})$. By Lemma 3, there exist two disjoint paths P_{uv} in $TQ_n^{0,0} \bigcup TQ_n^{1,0}$ and P_{xy} in $TQ_n^{0,1} \bigcup TQ_n^{1,1}$,respectively, of length $2^{n+1} - 1$. □

4 Conclusion

In this paper, we discussed the two-equal-disjoint path coverable problem and proved that Twisted Cubes TQ_n are globally two-equal-disjoint path coverable for $n \geq 5$. The globally two-equal-disjoint path coverable problem is a extension of Hamiltonian connected problem. We can see Hamiltonian connected problem as globally one-path coverable problem, and then we extended this property to globally two-equal-disjoint path coverable. This work may help to discuss the many-to-many disjoint path coverable problem.

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