Edge-disjoint Undirected Spanning Trees on the Wrapped Butterfly Networks

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Abstract

The problem of finding the maximum number of edge-disjoint spanning trees arises from the need for developing efficient collective communication algorithms in distributed memory parallel computers. In this paper, we propose a formula for obtaining the maximum number of edge-disjoint undirected spanning trees on the wrapped butterfly network. The result can be applied to design efficient multicast routing algorithms in wormhole-routed parallel systems.

Keywords: Interconnection network; Graph; Butterfly network; Spanning tree

1. Introduction

A multiprocessor/communication interconnection network is usually modeled as a graph, in which the vertices correspond to processors/nodes and the edges correspond to connections or communication links. Therefore we use the terms, graphs and networks, interchangeably. Among various kinds of popular network topologies, butterfly networks are very suitable for VLSI implementation and parallel computing. Recently, the wrapped butterfly graph has gained many researchers' efforts for its nice topological properties [2, 4, 6, 8, 10–12].

Embedding one network onto another is an interesting subject because the portability of the guest network onto the host network would permit executing guest specified algorithms on the host with as little modification as possible. Embedding various topologies, such as rings, linear arrays, binary trees, etc., into the butterfly networks has been addressed in research by [5, 11, 12]. In particular, the problem of constructing edge-disjoint spanning trees in a network arises from the need for developing efficient collective communication algorithms in distributed memory parallel computers. Barden et al. [1] presented a brief comparison between two routing schemes, *store-and-forward* routing [7] versus *wormhole* routing [9], and explained how and why edge-disjoint spanning trees are involved in these applications. Not only did Touzene et al. [10] investigate how to embed edge-disjoint directed spanning trees on butterfly networks, but they also discussed the possible applications to communication algorithms. Since the proposed spanning trees are directed, their construction permits an edge (u, v) to be used in orientation $\langle u, v\rangle$ in one spanning tree and in orientation $\langle v, u\rangle$ in a second spanning tree. Such kind of applications are mainly based on the store-and-forward routing. Unlike the previous research, we turn our attention to undirected spanning trees, which can be applied to the wormhole routing. In [1], a recursive method was presented to construct $\lfloor \frac{n}{2} \rfloor$ edge-disjoint undirected spanning trees on an n-cube. In this paper, we give a formula for obtaining the maximum number of edge-disjoint undirected spanning trees on the wrapped butterfly network.

The rest of this paper is organized as follows. In Section 2, we introduce graph-theoretic terminologies and the definition of wrapped butterfly networks. Section 3 is devoted to basic properties of the wrapped butterfly network. In Section 4, we show how to embed the maximum number of edge-disjoint undirected spanning trees onto the wrapped butterfly network. Finally, the concluding remarks are presented in Section 5.

2. Preliminaries

In this paper, we concentrate on loopless undirected graphs. For the notations and graph-theoretic terminologies, we follow the ones given by Bondy and Murty [3]. A graph G is a two-tuple (V, E), where V is a nonempty set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered

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pair of V}. We say that $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set*. Two vertices, u and v, are *adjacent* if $(u, v) \in E$. The number of vertices in a graph G, denoted by $|V(G)|$, is called the *order* of G; the number of edges, denoted by $|E(G)|$, is the *size* of G. The *degree* of any vertex u in a graph G, denoted by $deg_G(u)$, is the number of edges incident with u. The maximum and minimum degrees among the vertex set are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is *k-regular* if $\Delta(G) = \delta(G) = k$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of vertices of a graph G. The subgraph *induced* by S is the subgraph of G with the vertex set S and the edge set consisting of those edges that join two vertices in S. Analogously, the subgraph *generated* by a nonempty set $F \subseteq E(G)$ is the subgraph of G with the edge set F and the vertex set consisting of those vertices incident to at least one edge of F. If X is a subset of edges of graph G, then $G - X$ is the spanning subgraph of G obtained by deleting the edges of X from $E(G)$. Two graphs, G_1 and G_2, are *isomorphic* if there exists a bijection μ from $V(G_1)$ onto $V(G_2)$ such that $(u, v) \in E(G_1)$ if and only if $(\mu(u), \mu(v)) \in E(G_2)$. This bijection μ is called an *isomorphism*.

A *path* P of length k from vertex x to vertex y in a graph G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$ if $k \geq 1$. We also write P as $\langle x, P, y\rangle$ to emphasize its beginning and ending vertices. A path of length 0, consisting of a single vertex x, is denoted by $\langle x\rangle$. Let u and v be vertices in a graph G. We say that u is connected to v if G contains a path between u and v. The graph G itself is connected if u is connected to v for every pair u, v of vertices of G. A subgraph H of graph G is a *component* of G if H is a maximal connected subgraph of G. A *cycle* is a path with at least three vertices such that the first vertex is adjacent to the last one. In order to emphasize the vertex order on a cycle, a cycle C of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A *tree* is a connected graph without cycles. A *spanning tree* of a graph G is a spanning subgraph of G that is a tree. Let T be a tree rooted at vertex r. The *height* of T, denoted by $\mathbf{height}(T)$, is the length of the longest path among all the paths from root r to any other vertices of T. The following theorem characterizes a tree.

Theorem 1. *[3] Let G be a graph. Then G is a tree if and only if G is connected and* $|E(G)| = |V(G)| - 1$.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ denote the set of integers modulo n. The n-dimensional k-ary *wrapped butterfly network* (or *butterfly network* for short), denoted by $BF(k, n)$, is a graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_k^n$. Each of the $n \times k^n$ vertices is labeled by a two-tuple $\langle \ell, a_0 \ldots a_{n-1}\rangle$ with a level $\ell \in \mathbb{Z}_n$ and an n-digit radix-k string $a_0 \ldots a_{n-1} \in \mathbb{Z}_k^n$. The edge set of $BF(k, n)$ can be defined in terms of the following $2k$ generators, f_i and f_i^{-1} with $i \in \mathbb{Z}_k$:

$$\begin{aligned} & f_i(\langle \ell, a_0 \ldots a_{n-1}\rangle) \\ & = \langle (\ell+1)_{\bmod n}, a_0 \ldots a_{\ell-1} a_\ell^{(i)} a_{\ell+1} \ldots a_{n-1}\rangle, \end{aligned}$$

and

$$\begin{aligned} & f_i^{-1}(\langle \ell, a_0 \ldots a_{n-1}\rangle) \\ & = \langle (\ell-1)_{\bmod n}, a_0 \ldots a_{\ell-2} a_{\ell-1}^{(-i)} a_\ell \ldots a_{n-1}\rangle, \end{aligned}$$

where $a_\ell^{(i)} \equiv a_\ell + i \pmod{k}$. By definition, $BF(k, n)$ is $2k$-regular. It should be noticed that $BF(k, 2)$ is a multigraph. The *level* of vertex $\langle \ell, a_0 \ldots a_{n-1}\rangle$ is ℓ. An edge joining a level-ℓ vertex and a level-$(\ell+1)_{\bmod n}$ vertex is called a level-ℓ edge. Figure 1(a) depicts $BF(2, 3)$, and Figure 1(b) is an isomorphic structure of $BF(2, 3)$ with the replication of level-0 vertices to ease visualization.

3. Fundamental properties of $BF(k, n)$

Suppose that k and n are two integers greater than or equal to two. For any $\ell \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_k$, we use $BF_\ell^i(k, n)$ to denote a subgraph of $BF(k, n)$ induced by $\{\langle h, a_0 \ldots a_{n-1}\rangle \in V(BF(k, n)) \mid a_\ell = i\}$. It is easy to see that $BF_\ell^i(k, n)$ is isomorphic to $BF_\ell^j(k, n)$ for any $i, j \in \mathbb{Z}_k$. Moreover, $BF_{\ell_1}^i(k, n)$ is isomorphic to $BF_{\ell_2}^i(k, n)$ for any $\ell_1, \ell_2 \in \mathbb{Z}_n$. Obviously, $\{BF_\ell^i(k, n) \mid i \in \mathbb{Z}_k\}$ forms a partition of $BF(k, n)$. With this observation, Wong [12] proposed a *stretching* operation to obtain $BF_\ell^i(k, n)$ from $BF(k, n-1)$ when $n \geq 3$. More precisely, the stretching operation can be described as follows.

Let $i \in \mathbb{Z}_k$ and $\ell \in \mathbb{Z}_n$ for $n \geq 2$. Furthermore, let $\mathcal{G}_n$ denote the set of all subgraphs of $BF(k, n)$. Suppose that $G \in \mathcal{G}_n$. We define the following subsets of $V(BF(k, n+1))$ and $E(BF(k, n+1))$:

$$\begin{aligned} V_1 &= \{v_h^i \mid 0 \leq h < \ell, v_h \in V(G)\}, \\ V_2 &= \{v_{h+1}^i \mid \ell < h \leq n-1, v_h \in V(G)\}, \\ V_3 &= \{v_\ell^i \mid v_\ell \text{ is incident to} \\ &\qquad \text{a level-}(\ell-1)_{\bmod n} \text{ edge in } G\}, \\ V_4 &= \{v_{\ell+1}^i \mid v_\ell \text{ is incident to a level-}\ell \text{ edge in } G\}, \\ E_1 &= \{(v_h^i, v_{h+1}^i) \mid 0 \leq h < \ell, (v_h, u_{h+1}) \in E(G)\}, \\ E_2 &= \{(v_{h+1}^i, v_{h+2}^i) \mid h \geq \ell, (v_h, u_{h+1}) \in E(G)\}, \end{aligned}$$

and

$$\begin{aligned} E_3 &= \{(v_\ell^i, v_{\ell+1}^i) \mid v_\ell \text{ is incident to at least one} \\ &\qquad \text{level-}(\ell-1)_{\bmod n} \text{ edge and at least one} \\ &\qquad \text{level-}\ell \text{ edge in } G\} \end{aligned}$$

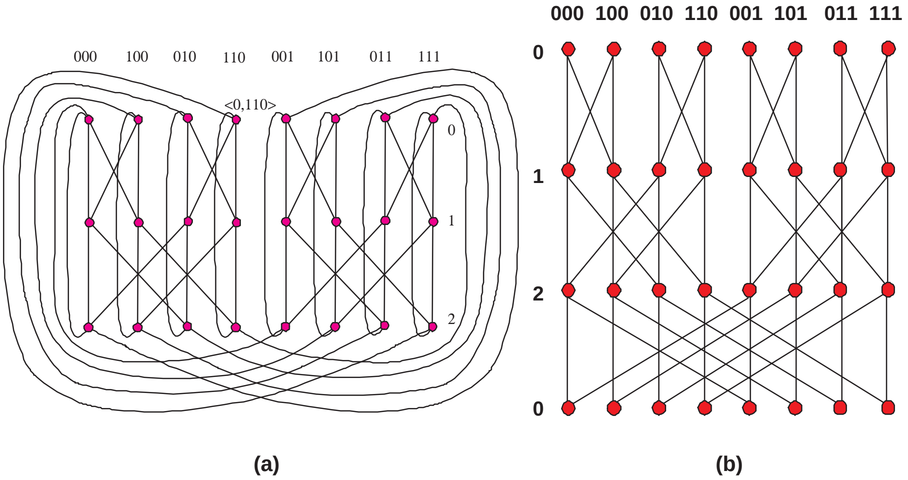

Figure 1. (a) The structure of $BF(2,3)$; (b) $BF(2,3)$ with level-0 vertices replicated to ease visualization.

where

$$\begin{aligned} v_h &= \langle h, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1} \rangle, \\ u_h &= \langle h, b_0 \ldots b_{\ell-1} b_\ell \ldots b_{n-1} \rangle, \\ v_h^i &= \langle h, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1} \rangle, \text{ and} \\ u_h^i &= \langle h, b_0 \ldots b_{\ell-1} i b_\ell \ldots b_{n-1} \rangle. \end{aligned}$$

Then the stretching function $\gamma_\ell^i : \bigcup_{n\geq 2} \mathcal{G}_n \to \bigcup_{n\geq 3} \mathcal{G}_n$ is defined by assigning $\gamma_\ell^i(G)$ as the graph with vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$ and with edge set $E_1 \cup E_2 \cup E_3$. Obviously, γ_ℓ^i is well-defined and one-to-one. Furthermore, $\gamma_\ell^i(G) \in \mathcal{G}_{n+1}$ if $G \in \mathcal{G}_n$. It is easy to see that $\gamma_\ell^i(BF(k,n)) = BF_\ell^i(k, n+1)$. In particular, we have $\gamma_\ell^i(P)$ is a path in $BF(k, n+1)$ if P is a path in $BF(k,n)$.

In the next lemma, we use the following notations:

$$\begin{aligned} v_\ell &= \langle \ell, a_0 \ldots a_{n-1} \rangle \text{ and} \\ v_\ell^i &= \langle \ell, i a_0 \ldots a_{n-1} \rangle. \end{aligned}$$

Lemma 1. *Suppose that G is a connected spanning subgraph of $BF(k,n)$ for $k \geq 2$ and $n \geq 3$. Let*

$$\begin{aligned} F_0 &= \{v_0 \in V(G) \mid v_0 \textit{ is not incident to any} \\ &\qquad \textit{level-}(n-1) \textit{ edge in } G\}, \\ F_1 &= \{v_0 \in V(G) \mid v_0 \textit{ is not incident to any} \\ &\qquad \textit{level-0 edge in } G\}. \end{aligned}$$

For $i \in \mathbb{Z}_k$, let

$$\begin{aligned} \overline{F_0^i} &= \{v_0^i \mid v_0 \in F_0\}, \\ \overline{F_1^i} &= \{v_1^i \mid v_0 \in F_1\}, \textit{ and} \\ M &= \bigcup_{v_0 \notin F_0 \cup F_1} \{(v_0^i, v_1^i)\}. \end{aligned}$$

Then $F_0 \cap F_1 = \emptyset$, $\overline{F_0^i} \cap \overline{F_1^i} = \emptyset$, $\overline{F_0^i} \cup \overline{F_1^i} = V(BF_0^i(k, n+1)) - V(\gamma_0^i(G))$, and $M \subseteq E(\gamma_0^i(G))$.

4. Edge-disjoint spanning trees of $BF(k,n)$

A reasonable upper bound on the number of edge-disjoint undirected spanning trees in $BF(k,n)$ is $\left\lfloor \frac{|E(BF(k,n))|}{|V(BF(k,n))|-1} \right\rfloor = \left\lfloor \frac{n \times k^{n+1}}{n \times k^n - 1} \right\rfloor = k$. In this section, we show that $BF(k,n)$ contains exactly k edge-disjoint undirected spanning trees.

Lemma 2. *Suppose $n \geq 2$ and $k \geq 2$. For every $i \in \mathbb{Z}_k$, let $\mathbf{s}_i = \langle 0, i^n \rangle$ be a vertex of $BF(k,n)$, and let G_i be a subgraph of $BF(k,n)$ generated by*

$$\bigcup_{t=0}^{n-1} \bigcup_{(p_0,\ldots,p_n) \in \{(q_0,\ldots,q_n) \in \mathbb{Z}_k^{n+1} \mid q_t = 0\}} \left\{ (u, f_{p_n}(u)) \mid u = f_0^t \circ f_{p_{n-1}} \circ \ldots \circ f_{p_0}(\mathbf{s}_i) \right\}.$$

Then $\{G_i \mid i \in \mathbb{Z}_k\}$ is a set of k spanning components of $BF(k,n)$ such that $\bigcup_{i=0}^{k-1} E(G_i) = E(BF(k,n))$, and $E(G_i) \cap E(G_j) = \emptyset$ whenever $i \neq j$. Moreover, $T_i = G_i - \{(f_0^{-1}(\mathbf{s}_i), \mathbf{s}_i)\}$ is a spanning tree of $BF(k,n)$ rooted at vertex $\mathbf{s}_i$.

Proof. Assume that $i \in \mathbb{Z}_k$. It is clear that every vertex of G_i is connected to $\mathbf{s}_i$. Thus G_i is connected. To see that G_i is a spanning component of $BF(k,n)$, we can decompose $E(G_i)$ into the following disjoint subsets. For any $0 \leq j \leq 2n-2$, let $F_{i,j} = \{(\mathbf{s}_i, f_p(\mathbf{s}_i)) \mid p \in \mathbb{Z}_k\}$ if $j = 0$; $F_{i,j} = \bigcup_{p \in \mathbb{Z}_k} \{(u, f_p(u)) \mid u$ is a level-j vertex incident to an edge in $F_{i,j-1}\}$ if $1 \leq$

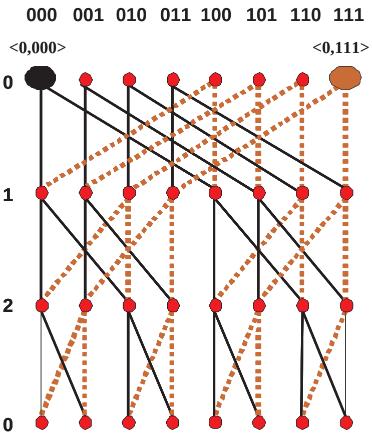

Figure 2. Two edge-disjoint undirected spanning trees of $BF(2,3)$, rooted at $\langle 0,000\rangle$ and $\langle 0,111\rangle$, respectively.

$j \le n-1$; $F_{i,j} = \bigcup_{(p_0,p_1,\ldots,p_n)\in\mathcal{X}}\{(u, f_{p_n}(u)) \mid u = f_0^{j-n}\circ f_{p_{n-1}}\circ\ldots\circ f_{p_0}(\mathbf{s}_i)\}$ if $n \le j \le 2n-2$, where $\mathcal{X} = \{(q_0,q_1,\ldots,q_n)\in\mathbb{Z}_k^{n+1} \mid q_{j-n} = 0, (q_{j-n+1},\ldots,q_{n-1}) \ne (0,\ldots,0)\}$. Then we observe that $E(G_i) = \bigcup_{j=0}^{2n-2} F_{i,j}$ and $F_{i,j}\cap F_{i',j'} = \emptyset$ whenever $i \ne i'$ or $j \ne j'$. Thus, $E(G_i)\cap E(G_{i'}) = \emptyset$ if $i \ne i'$. By counting, we have $|E(G_i)| = \left|\bigcup_{j=0}^{2n-2} F_{i,j}\right| = n\times k^n$. Since $|E(BF(k,n))| = n\times k^{n+1} = \sum_{t=0}^{k-1}|E(G_t)|$, we obtain $\bigcup_{t=0}^{k-1}E(G_t) = E(BF(k,n))$. Moreover, we have $|V(G_i)| = n\times k^n$. Hence G_i is indeed a spanning component of $BF(k,n)$. It is easy to see that $V(T_i) = V(G_i)$. Therefore, we obtain $|V(T_i)| = |V(G_i)| = n\times k^n$ and $|E(T_i)| = |E(G_i)| - 1 = n\times k^n - 1$. By Theorem 1, T_i turns out to be a spanning tree of $BF(k,n)$ rooted at vertex $\mathbf{s}_i$. Therefore the proof is completed. □

Example 1. *In Figure 2, we depict two edge-disjoint spanning trees of $BF(2,3)$, which are rooted at $\langle 0,000\rangle$ and $\langle 0,111\rangle$, respectively.*

Example 2. *In Figure 3, we depict three edge-disjoint spanning trees of $BF(3,3)$, which are rooted at $\langle 0,000\rangle$, $\langle 0,111\rangle$, and $\langle 0,222\rangle$, respectively.*

Since $BF(k,n)$ is vertex-transitive, we have the following corollary.

Corollary 1. *Suppose that $n,k \ge 2$. Let $\ell \in \mathbb{Z}_n$ and $a_0\ldots a_{n-1}\in\mathbb{Z}_k^n$. For any $i\in\mathbb{Z}_k$, let $\mathbf{r}_i = \langle \ell, a_0^{(i)}\ldots a_{n-1}^{(i)}\rangle$. Then there exist k edge-disjoint spanning trees of $BF(k,n)$ rooted at $\mathbf{r}_1,\ldots,\mathbf{r}_k$, respectively. Furthermore, each of these k edge-disjoint spanning trees has height $2n-1$.*

Theorem 2. *Let $\mathbf{r}$ be any vertex of $BF(k,n)$ with $k,n\ge 2$. Then $BF(k,n)$ contains k edge-disjoint undirected spanning trees rooted at $\mathbf{r}$, with k unused edges incident with $\mathbf{r}$. One of these k spanning trees has height $2n+1$, and the other $k-1$ spanning trees have height $2n$.*

Proof. Without loss of generality, we assume that $\mathbf{r} = \langle 0,0^n\rangle$. We partition $BF(k,n)$ into $\{BF_0^i(k,n)\mid i\in\mathbb{Z}_k\}$. For any $j\in\mathbb{Z}_k$, let $X_j = \{(\langle 0,jw\rangle, f_q(\langle 0,jw\rangle)) \mid w\in\mathbb{Z}_k^{n-1}\setminus\{0^{n-1}\}, q\in\mathbb{Z}_k\setminus\{0\}\}$ if $j=0$ and $X_j = \{(\langle 0,jw\rangle, f_q(\langle 0,jw\rangle)) \mid w\in\mathbb{Z}_k^{n-1}, q\in\mathbb{Z}_k\setminus\{0\}\}\cup\{(\mathbf{r}, f_j(\mathbf{r}))\}$ otherwise. Moreover, for any $j\in\mathbb{Z}_k$, let $Y_j = \bigcup_{t\ne j}\{(\langle 0,tw\rangle,\langle 1,tw\rangle) \mid w\in\mathbb{Z}_k^{n-1}\} - \bigcup_{t\ne j}\{(f_t(\mathbf{r}), f_0^{-1}(f_t(\mathbf{r})))\}$ if $j=0$; $Y_j = \{(f_j(\mathbf{r}), f_0^{-1}(f_j(\mathbf{r})))\}\cup\bigcup_{t\ne j}\{(\langle 0,tw\rangle,\langle 1,tw\rangle)\mid w\in\mathbb{Z}_k^{n-1}\}$ otherwise.

Suppose that $n=2$. We first construct k spanning components of $BF_0^i(k,2)$ with $i\in\mathbb{Z}_k$. For every $j\in\mathbb{Z}_k$, let $\Gamma_{i,j}$ be a subgraph of $BF_0^i(k,2)$ generated by $\{(\langle 0,iw\rangle,\langle 1,iw\rangle)\mid w\in\mathbb{Z}_k\}\cup\{(\langle 1,ij\rangle,\langle 0,ij^{(w)}\rangle)\mid w\in\mathbb{Z}_k\}$. Then we set T_p, with $p\in\mathbb{Z}_k$, to be the subgraph of $BF(k,2)$ generated by

$$\left(\bigcup_{j=0}^{k-1} E\left(\Gamma_{(p+j)\bmod k, j}\right)\cup X_p - Y_p\right) - \left\{\left(f_p^{-1}(\mathbf{r}),\mathbf{r}\right)\right\}.$$

Obviously, $\{T_0,\ldots,T_{k-1}\}$ is a set of edge-disjoint undirected spanning trees of $BF(k,2)$ rooted at $\mathbf{r}$. It is easy to see that the set of k unused edges is $\{(f_p^{-1}(\mathbf{r}),\mathbf{r})\mid p\in\mathbb{Z}_k\}$.

Suppose that $n\ge 3$. First of all, we use Lemma 2 to construct k edge-disjoint components $G_0,\ldots,G_{k-1}$ of $BF(k,n-1)$ such that $G_j - \{(f_0^{-1}(\mathbf{s}_j),\mathbf{s}_j)\}$, where $\mathbf{s}_j = \langle 0, j^{n-1}\rangle$, is a spanning tree of $BF(k,n-1)$ rooted at $\mathbf{s}_j$. Since $BF_0^i(k,n) = \gamma_0^i(BF(k,n-1))$ with $i\in\mathbb{Z}_k$, Lemma 1 ensures that $V(BF_0^i(k,n)) - V(\gamma_0^i(G_j)) = \{\langle 0,ixw\rangle \mid x\in\mathbb{Z}_k - \{j\}, w\in\mathbb{Z}_k^{n-2}\}$. Hence, let $\Gamma_{i,j}$ be a subgraph of $BF_0^i(k,n)$ generated by $E(\gamma_0^i(G_j))\cup\{(\langle 0,iw\rangle,\langle 1,iw\rangle)\mid w\in\mathbb{Z}_k^{n-1}\}$ for $j\in\mathbb{Z}_k$. Similarly, we set T_p, with $p\in\mathbb{Z}_k$, to be the subgraph of $BF(k,n)$ generated by $\left(\bigcup_{j=0}^{k-1}E\left(\Gamma_{(p+j)\bmod k,j}\right)\cup X_p - Y_p\right) - \left\{\left(f_p^{-1}(\mathbf{r}),\mathbf{r}\right)\right\}$. Then $\{T_0,\ldots,T_{k-1}\}$ forms a set of edge-disjoint undirected spanning trees of $BF(k,n)$ rooted at $\mathbf{r}$. Again, the set of k unused edges is $\{(f_p^{-1}(\mathbf{r}),\mathbf{r})\mid p\in\mathbb{Z}_k\}$. Clearly, we have $\mathbf{height}(T_p) = 2n+1$ if $p=0$ and $\mathbf{height}(T_p) = 2n$ otherwise.

The proof is completed. □

Example 3. *In Figure 4, we depict two edge-disjoint undirected spanning trees of $BF(2,3)$ rooted at $\langle 0,000\rangle$. In*

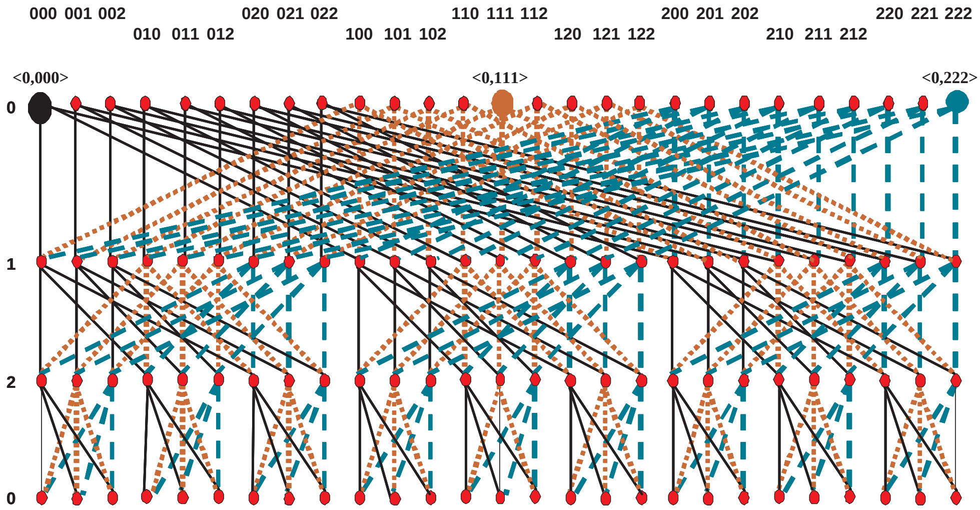

Figure 3. Three edge-disjoint undirected spanning trees of $BF(3,3)$, rooted at $\langle 0,000\rangle$, $\langle 0,111\rangle$, and $\langle 0,222\rangle$, respectively.

Figure 5, we illustrate three edge-disjoint undirected spanning trees of $BF(3,3)$ *rooted at* $\langle 0,000\rangle$.

5. Conclusion

In this paper, we show that every $BF(k,n)$ contains k edge-disjoint rooted spanning trees whose heights do not exceed $2n+1$, provided $k \geq 2$ and $n \geq 2$. Our result has applications to multicast communication in wormhole-routed parallel systems.

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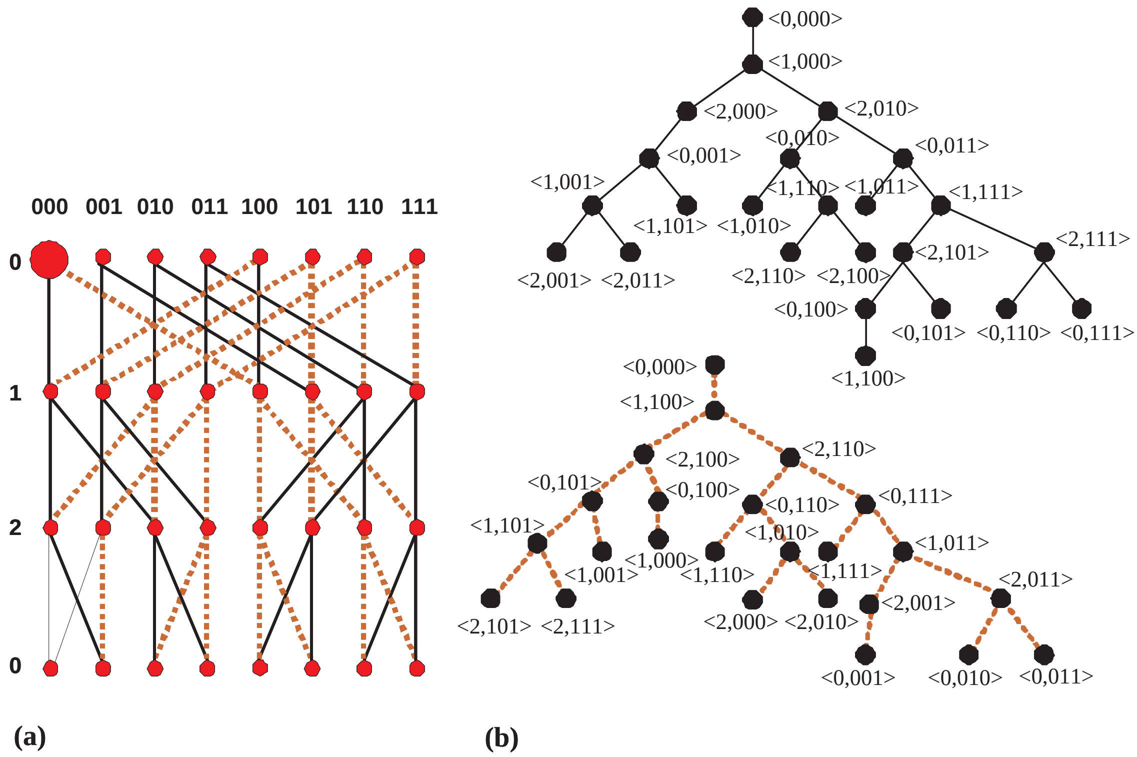

Figure 4. Two edge-disjoint undirected spanning trees of $BF(2,3)$ rooted at $\langle 0, 000\rangle$.

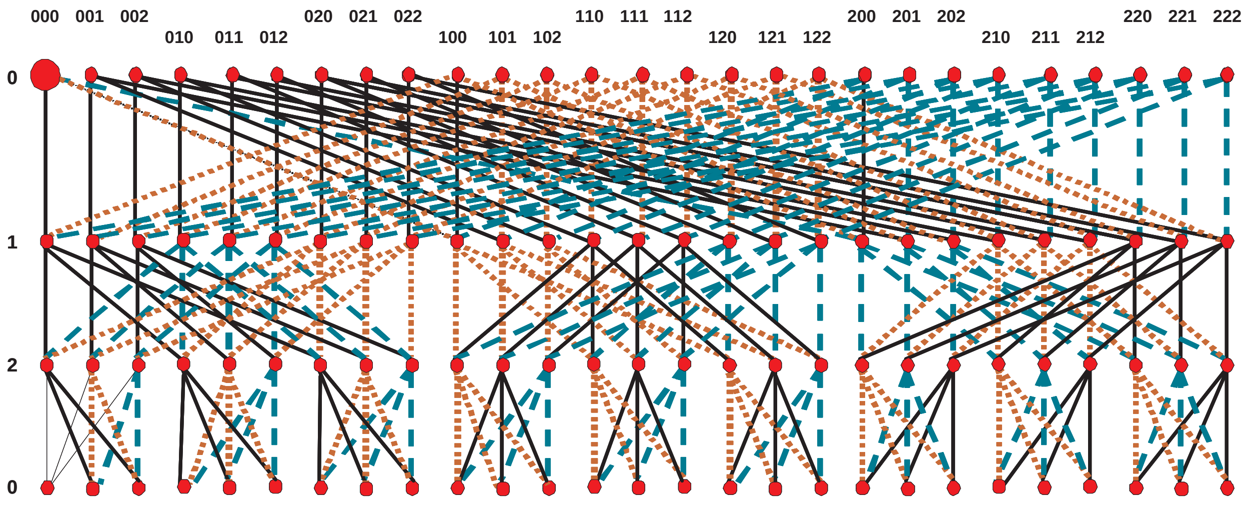

Figure 5. Three edge-disjoint undirected spanning trees of $BF(3,3)$ rooted at $\langle 0, 000\rangle$.