有向分割星狀網路的唯一最小支配集合
The Unique Minimum Dominating Set of Directed Split-Stars

王福星 張肇明 王有禮 許呈如

Fu–Hsing Wang[1] Jou–Ming Chang[2] Yue–Li Wang[1,*] Cheng–Ju Hsu[3]

[1]Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan, Republic of China.

[2]Department of Information Management, National Taipei College of Business, Taipei, Taiwan, Republic of China.

[3]Department of Computer Science, National Taiwan Ocean University, Keelung, Taiwan, Republic of China.

Abstract

A dominating set S of a directed graph D is a set of vertices such that every vertex not in S is dominated by at least one vertex of S. In this paper, we show that there is a unique minimum distance-k dominating set, for $k = 1, 2$, in a directed split-star, which has recently been developed as a new model of the interconnection network for parallel and distributed computing systems.

摘 要

在本篇論文中, 將證明在有向分割星狀網路中, 距離爲一及二的最小支配集合, 是唯一的一組集合。

Keyword: Dominating set, Interconnection network, Directed split-star.

*All correspondence should be addressed to Professor Yue–Li Wang, Department of Information Management, National Taiwan University of Science and Technology, 43, Section 4, Kee–Lung Road, Taipei, Taiwan, Republic of China. (Phone: 886–02–27376768, Fax: 886–02–27376777, Email: ylwang@cs.ntust.edu.tw).

1 Introduction

Let $D = (V, A)$ be a directed graph (digraph) with vertex set V and arc set A, where $A \subseteq V \times V$. An arc $< u, v >$ is said to be *directed* from u to v, in which case we say that u *dominates* v, and v is *dominated* by u. The *outset* of a vertex u in D is the set $O(u) = \{v \in V \mid < u, v > \in A\}$, while the *closed outset* is $O[v] = O(v) \cup \{v\}$. For a subset $S \subseteq V$, we also define $O(S) = \bigcup_{v \in S} O(v)$ and $O[S] = O(S) \cup S$. Let $od(v) = |O(v)|$ be the *outdegree* of a vertex $v \in V$, and $\Delta(D)$ the maximum outdegree among all vertices in the digraph D. Undirected graphs form in a sense a special class of digraphs, and an undirected edge (u, v) is a pair of arcs $<u, v>$ and $<v, u>$.

An *independent set* S of a digraph D is a set of vertices such that no two vertices of S are joined by an arc. A set $S \subseteq V$ is a *dominating set* of D if every vertex $v \in V \backslash S$ is dominated by at least one vertex of S, where $V \backslash S = \{v \in V \mid v \notin S\}$. A dominating set with the minimum number of vertices is called a *minimum dominating set*, and its cardinality, denoted by $\gamma(D)$, is termed the *domination number* of D. Researches on domination number and related parameters have much been attracted by graph theorists for their strongly practical applications and theoretical interesting. For instance, consider an interconnection network modeled by a graph, for which vertices represent processors and edges represent direct communication links between pairs of processor. Assume that from time to time we need to collect information from all processors, and this work must be done relatively often and fast. Thus we cannot route this information over too long a path. This suggests to identify a small set of processors (a dominating set) which are close to all other processors. It is well-know that the problem of finding a minimum dominating set is NP-complete for general undirected graphs and specifically for many restricted classes of graphs [3]. For a thorough treatment of domination in graphs, we refer the reader to [4, 5].

Although domination and related topics have been extensively studied, the respective analogs on digraphs have not received much attention. In this paper, we present results concerning domination in a special class of digraphs called directed split-stars, which has recently been developed as a new model of the interconnection network for parallel and distributed computing systems. Cheng et al. [2] gave a variant distributed processor architecture of the star graphs which is known as the split-stars. Cheng and Lipman [1] proposed an assignment of orientation to the split-stars and showed that the resulting digraphs are not only strongly connected, but, in fact, they have maximal arc-fault tolerance and a small diameter.

The *n-dimensional directed split-star* $\overrightarrow{S_n^2}$ is a directed graph whose vertices are in a one-to-one correspondence with $n!$ permutations $[p_1, p_2, \ldots, p_n]$ of the set $N = \{1, 2, \ldots, n\}$, and two vertices u, v of $\overrightarrow{S_n^2}$ are connected by an arc $<u, v>$ if and only if the permutation of v can be obtained from u by either a 2-exchange or a 3-rotation. Let $u = [p_1, p_2, \ldots, p_n]$. A 2-*exchange* interchanges the first symbol p_1 with the second symbol p_2 whenever $p_1 > p_2$, i.e., $v = [p_2, p_1, \ldots, p_n]$. A 3-*rotation* counterclockwise rotates the symbols in positions 1, 2 and i for some $i \in \{3, 4, \ldots, n\}$, i.e., $v = [p_i, p_1, p_3, \ldots, p_{i-1}, p_2, p_{i+1}, \ldots, p_n]$. Figure 1 depicts an example of $\overrightarrow{S_n^2}$ for $n = 4$..

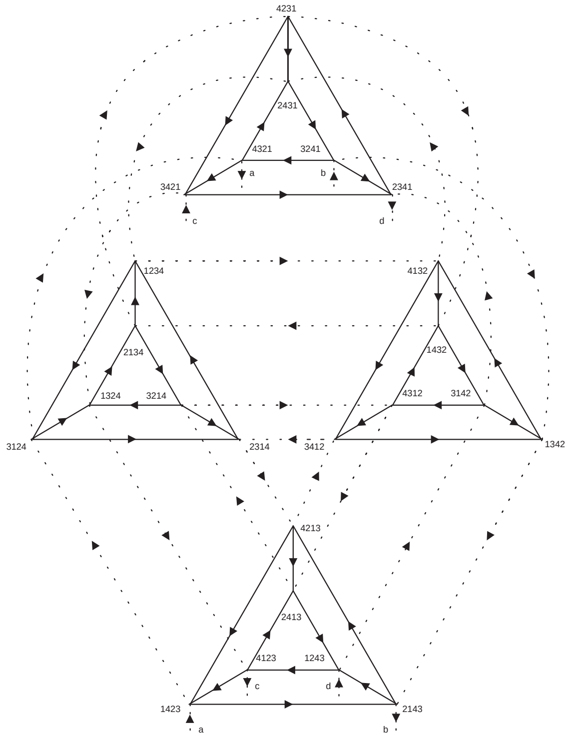

Figure 1: 4-dimensional directed split-star.

The remaining part of this paper is organized as follows. In Section 2, we study the problem of finding minimum domination set on directed split-stars. A result shows that $\overrightarrow{S_n^2}$ has a unique minimum dominating set of size $(n-1)!$. In Section 3, we further investigate the distance-2 domination in directed split-stars and obtain a similar result as the previous section. Finally, a concluding remark is given in the last section.

2 The Unique Minimum Dominating Set

Let $v = [p_1, p_2, \ldots, p_n]$. The 2-*exchange neighbor* of v is $E(v) = [p_2, p_1, \ldots, p_n]$, and the 3-*rotation neighbors* of v are $F^i(v) = [p_i, p_1, p_3, \ldots, p_{i-1}, p_2, p_{i+1}, \ldots, p_n]$ for $3 \leq i \leq n$.

Proposition 1 *Let u be a vertex of $\overrightarrow{S_n^2}$. If v is the 2-exchange neighbor of u, then $od(u) = n-1$ and $od(v) = n-2$.*

Proof. By definition, u has $n-2$ 3-rotation neighbors and one 2-exchange neighbor. Thus, $od(u) = n-1$. Since v has no 2-exchange neighbor, $od(v) = n-2$.

Q. E. D.

Since each vertex of $\overrightarrow{S_n^2}$ is either a 2-exchange neighbor of some vertex or a vertex having a 2-exchange neighbor, there are $\frac{n!}{2}$ vertices with the maximum outdegree in $\overrightarrow{S_n^2}$. In particular, $\Delta(\overrightarrow{S_n^2}) = n-1$. To simplify the description of our results, we define the following sets. Let $N_i = N \backslash \{i\}$ for $i = 1, 2$, and let $N_{12} = N \backslash \{1, 2\}$, where $N = \{1, 2. \ldots, n\}$.

Lemma 2 *Let $S = \{[p_1, 1, p_3, \ldots, p_n] \,|\, p_i \in N_1, i \in N_2\}$. Then, S is a minimum dominating set of $\overrightarrow{S_n^2}$, and $\gamma(\overrightarrow{S_n^2}) = (n-1)!$. In particular, every vertex not in S is dominated by exactly one vertex of S.*

Proof. It is clear that S is an independent set of $\overrightarrow{S_n^2}$ and $od(v) = \Delta(\overrightarrow{S_n^2}) = n-1$ for every vertex $v \in S$. Thus, the total outdegree of the vertices in S is $\sum_{v \in S} od(v) = n! - (n-1)!$. Since any vertex in $\overrightarrow{S_n^2}$ can dominate at most $\Delta(\overrightarrow{S_n^2})$ vertices, we have

$$\gamma(\overrightarrow{S_n^2}) \geq \frac{n!}{\Delta(\overrightarrow{S_n^2}) + 1} = (n-1)! = |S|.$$

Thus we only need to show that S is a dominating set of $\overrightarrow{S_n^2}$. Suppose that S is not a dominating set. We claim that there exists a vertex $r \in V \backslash S$ which is dominated by at least two vertices of S. This can easily be seen from the fact that the total outdegree of the vertices in S is equal to the number of vertices in $V \backslash S$ and there are no arcs between any two vertices in S.

Let $u = [u_1, 1, u_3, \ldots, u_n]$ and $v = [v_1, 1, v_3, \ldots, v_n]$ be any two distinct vertices in S and $r \in V \backslash S$ such that $r \in O(u) \cap O(v)$. Since r is dominated by u, either a 2-exchange or a 3-rotation exists from u to r. Assume that r is the 2-exchange neighbor of u, i.e., $r = [1, u_1, u_3, \ldots, u_n]$. Since the symbol 1 is on the second position of v, r cannot be a 3-rotation neighbor of v. This implies that r must be the 2-exchange neighbor of v and $u = v$, which contradicts that u and v are two distinct vertices of S. On the other hand, if r is a 3-rotation neighbor of u, then $r = F^i(u) = [u_i, u_1, u_3, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n]$ for some $i \in \{3, \ldots, n\}$. Since the symbol 1 is not on the first position of r, r cannot be the 2-exchange neighbor of v. So r must be also a 3-rotation neighbor of v. Since the symbol 1 is on the ith position of r, it implies $r = F^i(v) = [v_i, v_1, v_3, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_n]$. Thus $u = v$, which leads to a contradiction.

Q. E. D.

Theorem 3 *Let $S = \{[p_1, 1, p_3, \ldots, p_n] \,|\, p_i \in N_1, i \in N_2\}$. Then, S is the unique minimum dominating set of $\overrightarrow{S_n^2}$.*

Proof. Suppose that S' is a minimum dominating set of $\overrightarrow{S_n^2}$ and $S' \neq S$. Then there exists a vertex $u = [u_1, 1, u_3, \ldots, u_n]$ of S such that $u \notin S'$. Since S' is a dominating set, u must be either a 2-exchange neighbor or a 3-rotation neighbor of some vertex $v \in S'$. However, u cannot be the 2-exchange neighbor of some vertex in S' since the symbol 1 is on the second position of u and $u_1 > 1$. Let $u = F^i(v)$ where $v \in S'$ and $i \in \{3, \ldots, n\}$. Then $v = [1, u_i, u_3, \ldots, u_{i-1}, u_1, u_{i+1}, \ldots, u_n] = E([u_i, 1, u_3, \ldots, u_{i-1}, u_1, u_{i+1}, \ldots, u_n])$. By Proposition 1, $od(v) = n-2$. Recall that $\sum_{w \in S} od(w) = |S| \cdot (n-1) = n! - (n-1)!$. Since $|S'| = |S|$ and $v \in S'$, the total outdegree of the vertices in S' is at most $n! - (n-$

$1)!-1$ and $|V\backslash S'| = n!-(n-1)!$. This contradicts to the assumption that S' is a dominating set. Therefore, the result directly follows from Lemma 2.

Q. E. D.

3 The Unique Minimum Distance-2 Dominating Set

A set S of vertices in a directed graph $D = (V, A)$ is called a *distance-*2 *dominating set* if every vertex in $V\backslash S$ is within distance 2 from some vertex of S. A distance-2 dominating set with the minimum number of vertices is called a *minimum distance-*2 *dominating set*, and its cardinality is denoted by $\gamma_2(D)$. The *distance-*2 *outset* of a vertex $v \in V$ is given by $O_2(v) = \bigcup_{w\in O(v)} O(w)$. If $S \subseteq V$ then $O_2(S) = \bigcup_{v\in S} O_2(v)$. Obviously, a set S is a distance-2 dominating set if and only if $O[S] \cup O_2(S) = V$.

Let S be a set of vertices in a directed split-star $\overrightarrow{S_n^2}$. We denote by $X(S)$ and $R(S)$ the sets consisting of the 2-exchange neighbors of S and the 3-rotation neighbors of S, respectively. That is, $X(S) = \{E(v) \,|\, v \in S\}$ and $R(S) = \{F^i(v) \,|\, v \in S, 3 \le i \le n\}$. In order to give a simple representation of $O_2(S)$, we use the following notation: $RX(S) = R(X(S))$, $XR(S) = X(R(S))$, and $RR(S) = R(R(S))$. Note that $RX(S) \cup XR(S) \cup RR(S) = O_2(S)$.

Lemma 4 *Let* $S = \{[2, 1, p_3, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}\}$*. The following statements are true:*

Proof. By definition,

$X(S) = \{[1, 2, p_3, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}\}$ and

$R(S) = \{[p_i, 2, p_3, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}, 3 \le i \le n\}$.

Clearly, $X(S) \cap R(S) = \emptyset$. Moreover, $O(S) = X(S) \cup R(S) = \{[p_1, 2, p_3, \ldots, p_n] \,|\, p_i \in N_2, i \in N_2\}$, i.e., $O(S)$ contains all vertices with the symbol 2 on the second position. So $O(S) \cap S = \emptyset$. Also we have

$RX(S) = \{[p_i, 1, p_3, \ldots, p_{i-1}, 2, p_{i+1}, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}, 3 \le i \le n\}$

and

$XR(S) = \{[2, p_i, p_3, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}, 3 \le i \le n\}$.

In addition, $RR(S)$ can be partitioned into the following three sets (where the position of the second 3-rotation is considered):

$RR_1(S) = \{[p_j, p_i, p_3, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_{j-1}, 2, p_{j+1}, \ldots, p_n] | p_3, \ldots, p_n \in N_{12}, 3 \le i < j \le n\}$,

$RR_2(S) = \{[p_j, p_i, p_3, \ldots, p_{j-1}, 2, p_{j+1}, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}, 3 \le j < i \le n\}$,

and

$RR_3(S) = \{[1, p_i, p_3, \ldots, p_{i-1}, 2, p_{i+1}, \ldots, p_n] \,|\, p_3, \ldots, p_n \in N_{12}, 3 \le i \le n\}$.

Furthermore, $O_2(S)$ is divided into five disjoint subsets $XR(S)$, $RX(S)$, $RR_1(S)$, $RR_2(S)$ and $RR_3(S)$. This shows that the statement (1) holds. Since there are no vertices of $O_2(S)$ having the symbols 2 and 1 in the foremost two positions, $O_2(S) \cap S = \emptyset$. Similarly, we can verify $O_2(S) \cap O(S) = \emptyset$ since $O_2(S)$ contains no vertices with the symbol 2 on the second position. Therefore, statement (2) holds. We now prove statement (3) by contradiction as follows. Let u and v be any two distinct vertices in $O(S)$ and assume $w \in O(u) \cap O(v)$. We consider the following cases.

Case 1: $w \in RX(S)$. In this case, u and v are 2-exchange neighbors of some vertices in S. Let $u = [1, 2, u_3, \ldots, u_n]$ and $v = [1, 2, v_3, \ldots, v_n]$. Assume that $w = F^i(u) = F^j(v)$ for some $i, j \in \{3, \ldots, n\}$. Since $F^i(u) = [u_i, 1, u_3, \ldots, u_{i-1}, 2, u_{i+1}, \ldots, u_n]$ and $F^j(v) = [v_j, 1, v_3, \ldots, v_{j-1}, 2, v_{j+1}, \ldots, v_n]$, it implies that $i = j$ and $u_k = v_k$ for $3 \le k \le n$. This contradicts that u and v are two distinct vertices in $O(S)$.

The following cases consider that u and v

are 3-rotation neighbors of some vertices in S, and let $u = [u_i, 2, u_3, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n]$ and $v = [v_j, 2, v_3, \ldots, v_{j-1}, 1, v_{j+1}, \ldots, v_n]$.

Case 2: $w \in XR(S)$. In this case, $w = E(u) = E(v)$. Thus, $[2, u_i, u_3, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n] = [2, v_j, v_3, \ldots, v_{j-1}, 1, v_{j+1}, \ldots, v_n]$. This implies $i = j$ and $u_k = v_k$ for $3 \leq k \leq n$. It again reaches a contradiction that u and v are two distinct vertices.

Case 3: $w \in RR_1(S)$. In this case, there exist i' and j' with $i < i'$ and $j < j'$ such that $w = F^{i'}(u) = F^{j'}(v)$. Since $F^{i'}(u) = [u_{i'}, u_i, u_3, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_{i'-1}, 2, u_{i'+1}, \ldots, u_n]$ and $F^{j'}(v) = [v_{j'}, v_j, v_3, \ldots, v_{j-1}, 1, v_{j+1}, \ldots, v_{j'-1}, 2, v_{j'+1}, \ldots, v_n]$, it implies that $i = j$, $i' = j'$, and $u_k = v_k$ for $3 \leq k \leq n$, which is a contradiction.

Case 4: $w \in RR_2(S)$. By considering $i' < i$ and $j' < j$, a proof similar to that of case 3 gives a contradiction.

Case 5: $w \in RR_3(S)$. In this case, $w = F^i(u) = F^j(v)$. Thus, $[1, u_i, u_3, \ldots, u_{i-1}, 2, u_{i+1}, \ldots, u_n] = [1, v_j, v_3, \ldots, v_{j-1}, 2, v_{j+1}, \ldots, v_n]$. This implies $i = j$ and $u_k = v_k$ for $3 \leq k \leq n$, a contradiction.

Q. E. D.

Lemma 5 *Let* $S = \{[2, 1, p_3, \ldots, p_n] \mid p_3, \ldots, p_n \in N_{12}\}$. *Then,* S *is a minimum distance-*2 *dominating set of* $\overrightarrow{S_n^2}$.

Proof. Clearly, $|S| = (n-2)!$. Recall that $X(S) \cap R(S) = \emptyset$, and $O(S) = X(S) \cup R(S)$ contains all vertices with the symbol 2 on the second position. Thus, $|O(S)| = (n-1)!$, $|X(S)| = |S| = (n-2)!$, and $|R(S)| = (n-1)! - (n-2)!$. By Lemma 4, the sets S, $O(S)$, and $O_2(S)$ are pairwise disjoint. Hence, for proving S is a distance-2 dominating set, it suffices to show that $O_2(S)$ contains exactly $n! - (n-1)! - (n-2)!$ vertices.

For each vertex $v \in X(S)$, it has the form $[1, 2, p_3, \ldots, p_n]$. Thus v contains $n-2$ 3-rotation neighbors. For each vertex $v \in R(S)$, it has the form $[p_i, 2, p_3, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n]$. Since $p_i > 2$, v has $n-2$ 3-rotation neighbors and one 2-exchange neighbor. Also we have shown in Lemma 4 that for any two vertices $u, v \in O(S)$, $O(u) \cap O(v) = \emptyset$. Thus, $|RX(S)| = (n-2) \cdot (n-2)!$, $|XR(S)| = |R(S)| = (n-1)! - (n-2)! = (n-2) \cdot (n-2)!$, and $|RR(S)| = (n-2)^2 \cdot (n-2)!$. In particular, $RX(S)$, $XR(S)$, and $RR(S)$ are pairwise disjoint. Therefore,

$$\begin{aligned} |O_2(S)| &= |RX(S)| + |XR(S)| + |RR(S)| \\ &= (n-2) \cdot (n-2)! + \\ &\quad (n-2) \cdot (n-2)! + \\ &\quad (n-2)^2 \cdot (n-2)! \\ &= n! - (n-1)! - (n-2)! \end{aligned}$$

Let $\Delta_2(\overrightarrow{S_n^2}) = \max_{v \in V} |O(v)| + |O_2(v)|$. By Proposition 1, it is easy to verify that for any vertex $v \in V$, $O_2(v)$ contains at most $(n-2) \cdot (n-1) + 1 \cdot (n-2) = n^2 - 2n$ vertices. Thus $\Delta_2(\overrightarrow{S_n^2}) = (n-1) + (n^2 - 2n) = n^2 - n - 1$. Consequently,

$$\gamma_2(\overrightarrow{S_n^2}) \geq \frac{n!}{\Delta_2(\overrightarrow{S_n^2}) + 1} = (n-2)! = |S|.$$

This shows that S is a minimum distance-2 dominating set of $\overrightarrow{S_n^2}$.

Q. E. D.

Now, we can state our main result as follows.

Theorem 6 *Let* $S = \{[2, 1, p_3, \ldots, p_n] \mid p_3, \ldots, p_n \in N_{12}\}$. *Then,* S *is the unique minimum distance-*2 *dominating set of* $\overrightarrow{S_n^2}$.

Proof. Suppose that S' is a minimum distance-2 dominating set of $\overrightarrow{S_n^2}$ and $S' \neq S$. Clearly, $|S'| = |S| = (n-2)!$ and $|O(S')| \leq \sum_{w \in S'} od(w) \leq |S'| \cdot (n-1) = (n-1)!$. Thus, $V \backslash O[S']$ contains at least $n! - (n-1)! - (n-2)!$ vertices. Also, we have shown that for any vertex $w \in V$ the number of vertices contained in $O_2(w)$ is at most $n^2 - 2n$. An upper bound of the total outdegree of vertices in $O(S')$ can be computed as follows:

$$\begin{aligned} \sum_{w \in O(S')} od(w) &\leq |S'| \cdot (n^2 - 2n) \\ &= n! - (n-1)! - (n-2)!. \end{aligned}$$

Since S' is a distance-2 dominating set, the two hand sides of the above inequality are equal. In particular, the outdegree of every vertex contained in $O[S']$ is indeed $n-1$.

Since $S' \neq S$, there exists a vertex $u = [2, 1, u_3, \ldots, u_n]$ of S such that $u \notin S'$. Since S' is a distance-2 dominating set, u must be contained in one of the sets $X(S')$, $R(S')$, $RX(S')$, $XR(S')$, and $RR(S')$. However, u cannot be a 2-exchange neighbor of some vertex since the symbol 2 comes before symbol 1 in u. Hence, $u \notin X(S')$ and $u \notin XR(S')$. This implies that there exists a vertex $v \in O[S']$ such that $u = F^i(v)$ where $i \in \{3, \ldots, n\}$. Thus, $v = [1, u_i, u_3, \ldots, u_{i-1}, 2, u_{i+1}, \ldots, u_n] = E([u_i, 1, u_3, \ldots, u_{i-1}, 2, u_{i+1}, \ldots, u_n])$. By Proposition 1, $od(v) = n-2$, which is a contradiction to the above argument.

Q. E. D.

4 Concluding Remark

Cheng et al.[1] gave an orientation to the split-stars, and showed that the oriented graphs $\overrightarrow{S_n^2}$ are maximally arc-connected and arc-fault tolerant. In this paper, we show that there is a unique minimum distance-k dominating set, for $k = 1, 2$, in a directed split-star. However, the problem of finding a minimum distance-k dominating set for $k \geq 3$ on directed split-stars is still unsolvable. Further, a natural question to ask is whether the distance-k dominating set for $k \geq 3$ is unique.

References

[1] Eddie Cheng and Marc J. Lipman, Orienting split-stars and alternating group graphs, *Networks*, Vol. **35**, pp. 139-144, 2000.

[2] Eddie Cheng, Marc J. Lipman and H.A. Park, An attractive variation of the star graphs: split-stars, Technical Report No. 98-3, Oakland University, 1998.

[3] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman, San Francisco, 1979.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[6] J.B. Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2001.