Feedback Vertex Set in Split-Stars and Alternating Groups

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Abstract

The feedback vertex set *F* of a graph *G* is a subset of vertices such that the removal of *F* from *G* induces an acyclic subgraph.

In this paper, we study the feedback vertex set problem on the directed and undirected spilt-stars and alternating group graphs separately. We give upper and lower bounds to the minimum feedback vertex set on the *n*-dimensional spilt-stars and *n*-dimensional alternating group graphs.

Keyword：Feedback vertex set, Interconnection network, Split-stars, Alternating Groups.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$, where $E \subseteq V \times V$. We also let $D = (V, A)$ be a directed graph with vertex set $V(D)$ and arc set $A(D)$, where $A \subseteq V \times V$. An arc $\overrightarrow{uv}$ is said to be *directed* from u to v. In a graph G (V, E), a *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively in the circle. Also, a graph with no cycle is called *acyclic*. The *feedback vertex set* F of a digraph $D = (V, A)$ is a subset of vertices $\overline{V} \subseteq V$ whose removal from D, induces an acyclic subgraph $D' = (V', A')$ where $V' = V \setminus \overline{V}$ and $A' = \{< u, v > \in \mathrm{A} \mid u, v \in V'\}$. A *feedback vertex set* with the minimum cardinality is called *minimum feedback vertex set*, and its cardinality denoted by $\mu(D)$. The feedback vertex set problem originated from applications in combinatorial circuit design, but have found their way into numerous other applications, such as deadlock prevention in operating system, constraint satisfaction, Bayesian inference in artificial intelligence, and graph theory. As an example, consider an interconnection network modeled by a graph, for which vertices represent processors and each edge $< i, j>$ represents the request of processor i for a resource allocated to a processor j. If there is a cycle in such a graph, a deadlock occurs and every processor in the cycle will wait for the requested resource and will never release the resource already allocated to it. In order to solve the deadlock, one can remove some processors from the graph and put them in a waiting queue. Therefore, it is clear that we want to minimize the number of processors removed and make the graph acyclic. It is well known that the problem of finding a minimum feedback vertex set is NP-hard for general networks [4], but there exits polynomial solutions for particular graphs [6, 7, 8, 9,10]. In order to obtain polynomial solutions, one can restrict these problems to special classes of graphs, such as interval graphs, permutation graphs, etc.

In this paper, we present results concerning feedback vertex set problem in both directed and undirected spilt-stars and alternating group graphs , which have recently been developed as a new model of the interconnection network for parallel and distributed computing systems. Jwo et al. [5] studied the alternating group graphs; Cheng et al. [2] studied a variant distributed processor architecture of the star graphs, which is known as the spilt-star. Cheng and Lipman [1] proposed an assignment of directions to the edges of the spilt-stars and the alternating groups. They also showed that resulting directed graphs are not only strongly connected, but, in fact, they have maximally arc-connected and have small diameters.

The *n-dimensional directed spilt-star* $\overrightarrow{S_n^2}$ is a directed graph, which has the set of $n!$ permutations of an n-set as the vertex set. The vertices of the spilt-stars are in a one-to-one correspondence with $n!$ permutations $[p_1,p_2,\ldots,p_n]$ of the set $N=\{1,2,\ldots,n\}$, and two vertices u, v of $\overrightarrow{S_n}$ are connected by an arc $< u, v >$ if and only if the permutation of v can be obtained from u by either a 2-exchange or 3-rotation. Let $u = [p_1,p_2,\ldots,p_n]$. A 2-exchange interchanges the first symbol p_1 with the second symbol p_2 whenever $p_1 > p_2$, i.e., $v = [p_2,p_1,\ldots,p_n]$. A 3-rotation rotates the symbols in positions 1, 2 and i for some $i \in \{3, 4,\ldots, n\}$, i.e., $v = [p_i,p_1,\ldots p_{i\text{-}1}, p_2, p_{i+1},\ldots, p_n]$. Figure 1 depicts an example of $\overrightarrow{S_n^2}$ for $n = 4$. On the other hand we also denote the *n-dimensional undirected spilt-star* by S_n. The undirected spilt-star can be obtained form directed spilt-stars by letting each arc with bi-direction. For simplicity, we discard the arc's directions of undirected spilt-stars, and Figure 2 depicts an example of S_n for $n = 4$.

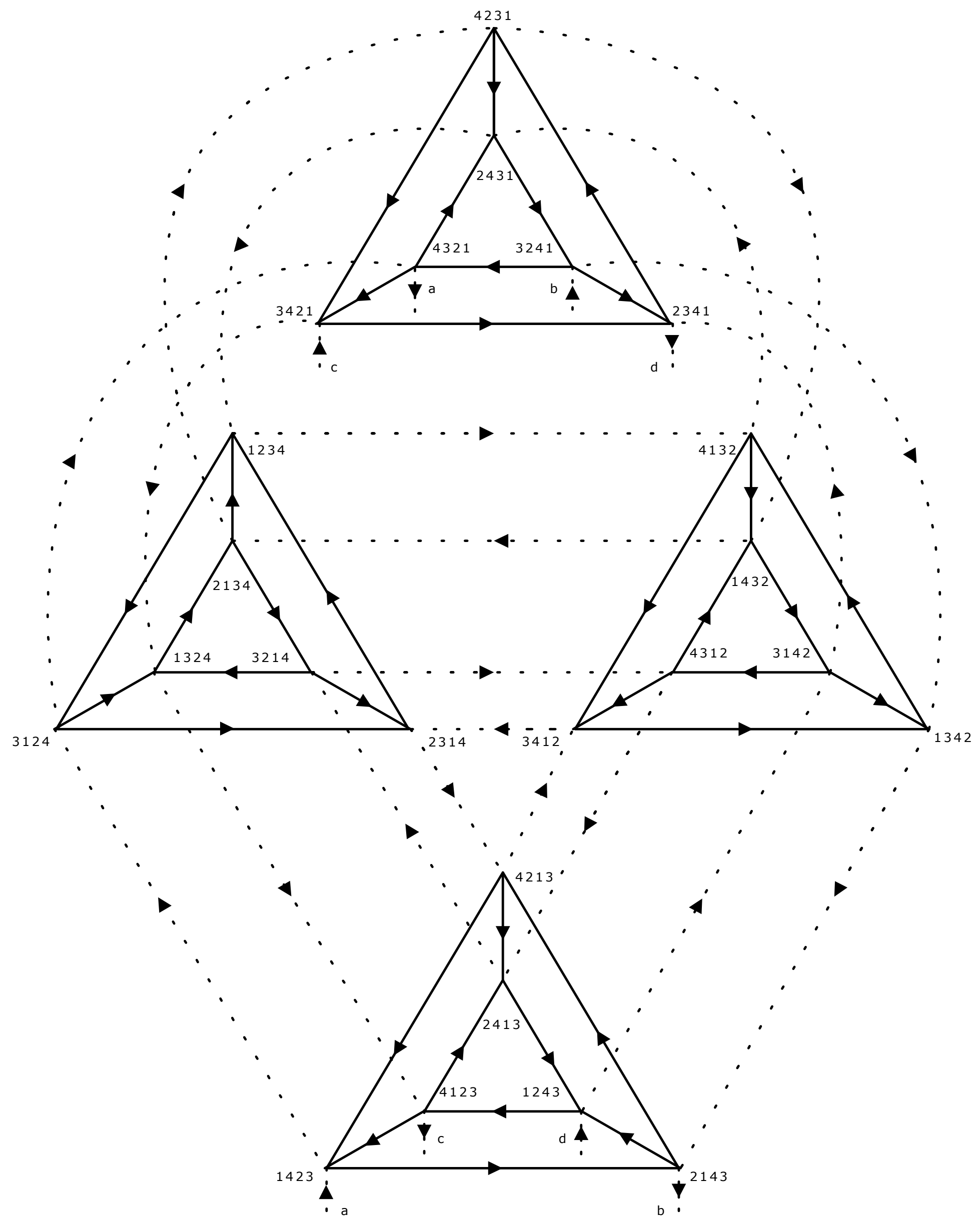

Figure 1 : 4-dimensional directed spilt-star.

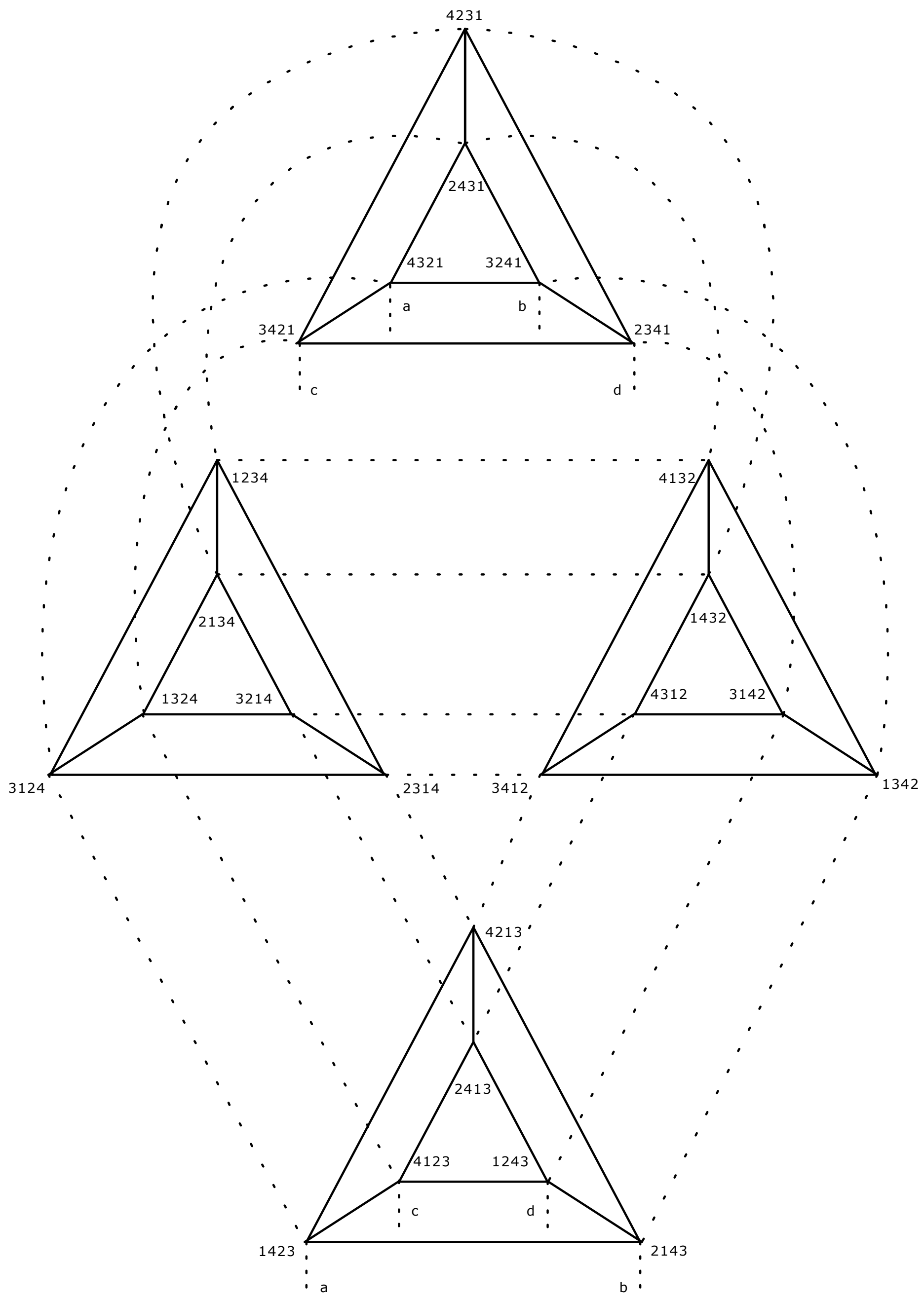

Figure 2 : 4-dimensional undirected spilt-star.

Let $u = [p_1, p_2, \ldots, p_n]$ where $p_i \in N$ for all $1 \leq i \leq n$. Now, let I_d denote the identity permutation of N. If $p_i > p_j$ where $i < j$, the pair p_i and p_j constitute an inversion. A permutation is said to be even (resp. odd) if its number of parity inversions is even (resp. odd). Given a simple graph G and a simple graph H, an *isomorphism* from G to H is a bijection $f: V(G) \rightarrow V(H)$ such that $\overline{uv} \in E(G)$ if and only if $f(u)\, f(v) \in E(H)$. We say G is isomorphic to H, if there is an isomorphism from G to H. Let $\overrightarrow{S_n^2,_E}$ be the subgraph of $\overrightarrow{S_n^2}$ induced by the set of even permutations. This is precisely the alternating group graph, $\overrightarrow{A_n}$, introduced in [5]. Let $\overrightarrow{S_n^2,_O}$ be the subgraph of $\overrightarrow{S_n^2}$ induced by the set of odd permutations. Then, $\overrightarrow{S_n^2,_E}$ is isomorphic to $\overrightarrow{S_n^2,_O}$ via a 2-exchange. Let A_n and $\overrightarrow{A_n}$ be *n-dimensional undirected* and *directed alternating group graphs* induced subgraph of S_n and $\overrightarrow{S_n^2}$ with even permutations reactively.

The remaining sections of this paper are organized as follows. In Section 2, we define some notations and study the feedback vertex set problem for the directed spilt-star. The upper and lower bounds to the minimum cardinality of the feedback vertex set for the n-dimensional directed spilt-star are given. In Section 3, we show the upper and lower bounds to the minimum cardinality of the feedback vertex set for the n-dimensional directed alternating group graphs. Section 4 and Section 5 are devoted to explore the existence of independent set vertices of spilt-stars and alternating group graphs to construct double rooted star as feedback vertex set of S_n and A_n. Finally, a concluding remark is given in the last section.

2. The Feedback Vertex Set of Directed Spilt-stars

The n dimensional spilt-star is a regular graph with degree $2n-3$, $|V(\overrightarrow{S_n^2})| = n!$ and $|E(\overrightarrow{S_n^2})| = (2n-3)n!/2$. $\overrightarrow{S_n^2}$ is recursively constructed by n copies of $\overrightarrow{S_{n-1}^2}$.

Let $N_i = N \setminus \{i\}$ for $i = 1,2$, and let $N_{1,2} = N \setminus \{1,2\}$, where $N = \{1,2,3\ldots n\}$. We also let $V(\overrightarrow{S_n}) = \{[p_1 p_2 p_3 \ldots p_n] \mid p_i \neq p_j \text{ and } i, j \in N\}$. We define X be a nonempty proper subset of the $V(\overrightarrow{S_n})$, and let $E_x(X)$ to be the set of 2-exchange neighbors of X and $R(X)$ to be the set of 3-rotation neighbors of X. Define $\delta(X)$ to be the set of arcs leaving X and $\rho(X)$ to be the set of arcs entering X.

Lemma 1. *Let* $F = \{[p_1, p_2, p_3 ... p_n] \mid p_1 > p_2 \}$ *and* $F \subseteq V(\overrightarrow{S_n})$*. Then F is a feedback vertex set of* $\overrightarrow{S_n}$.

Proof. Let F be a subset of $V(\overrightarrow{S_n})$ with cardinality $n!/2$. We want to show that F is a feedback vertex set of $\overrightarrow{S_n}$. Suppose, to the contrary, that F is not a feedback vertex set of $\overrightarrow{S_n}$.

Then a cycle $C = u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_k \rightarrow u_1$ exists in $\overrightarrow{S_n} \backslash F$. For each vertex $u_i = [u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,n}] \in C$. Since $u_{i,1} < u_{i,2}$, u_i has no 2-exchange neighbor. Therefore, u_{i+1} is a 3-rotation neighbor of u_i, $1 \le i \le k-1$, and u_1 is u_k's 3-rotation neighbor. Further, $u_{i,1} = u_{i+1,2}$ and $u_{k,1} = u_{1,2}$. Since $u_{i+1,1} < u_{i+1,2}$, $u_{i+1,1} < u_{i,1}$. Thus, $u_{k,1} < u_{k-1,1} < \ldots < u_{11} < u_{k,1}$, which is a contradiction. □

Lemma 2. *Let* $F' = \{[2,1,p_3,\ldots,p_n] \mid p_i \in N_{1,2}$, for each $3 \le i \le n\}$ *and* $F' \subseteq F$*. For each vertex* $u \in F'$, $R(u) \subseteq F \backslash F'$ *and* $R(E_x(u)) \subseteq F \backslash F'$.

Proof. $F' \subseteq V(\overrightarrow{S_n})$. Let $\overrightarrow{uv} \in \delta(u)$, then either $v \in E_x(u)$ or $v \in R(u)$. If $v \in R(u)$, it has the form $[p_i, 2, p_3, .. p_{i-1}, 1, p_{i+1}, \ldots, p_n]$ and $p_i \ge 2$, $i \in N_{1,2}$. Since $R(u) \in F$ and $R(u) \notin F'$, $R(u) \subseteq F \backslash F'$. Otherwise, if $v \in E_x(u)$ then v is the form of $[1,2,p_3,\ldots,p_n]$. Since $E_x(u) \not\subset F$, $E_x(u) \not\subset F \backslash F'$,but for each vertex $w \in R(E_x(u))$, it has the form of $[p_i, 1, p_3, .. p_{i-1}, 2, p_{i+1} \ldots, p_n]$ and $p_i \ge 1$, $p_i \ne 2$, $i \in N_{1,2}$. . Thus, $R(E_x(u)) \notin F'$, $R(E_x(u)) \not\subset F \backslash F'$. □

Lemma 3. *Let* $F'' = \{[n, n-1, p_3, \ldots, p_n] \mid p_i \in N_{n,n-1}$, for each $3 \le i \le n\}$ *and* $F'' \subseteq F$*. For each vertex* $u \in F''$, $R(u) \subseteq F \backslash F''$.

Proof. $F'' \subseteq V(\overrightarrow{S_n})$. Let $\overrightarrow{vu} \in \rho(u)$, then $v \in R(u)$. If $v \in R(u)$, it has the form $[n-1, p_i, p_3, .. p_{i-1}, n, p_{i+1}, \ldots, p_n]$ and $p_i \le n-1$, $i \in N_{1,2}$. Since $R(u) \in F$ and $R(u) \notin F''$, $R(u) \subseteq F \backslash F''$. □

Based on the Lemmas 1, 2 and 3, we give the following algorithm to find the feedback vertex set for the directed split-stars.

Algorithm FDS

Input: A directed split-star $\overrightarrow{S_n}$.

Output: A feedback vertex set of $\overrightarrow{S_n}$.

Method:

Step 1: $F = \{[p_1, p_2, p_3 \dots p_n] \mid p_1 > p_2\}$

$F' = \{[2,1,p_3,\dots,p_n] \mid p_i \in N_{1,2}$, for each $3 \le i \le n\}$

$F'' = \{[n,n-1,p_3,\dots,p_n] \mid p_i \in N_{n,n-1}$, for each $3 \le i \le n\}$

Step 2: $S = F / (F' \cup F'')$.

Step 3: **output** S.

From Lemma 1, Lemma 2 and Lemma 3, we can conclude that the upper bound to the minimum cardinality of the feedback vertex set for the n-dimensional directed spilt-star.

Theorem 4. $\mu(\overrightarrow{S_n}) \le n!/2 - 2(n-2)!$

Proof. For each vertex $u \in F'$, by Lemma 2, $R(u) \subseteq F \setminus F'$. Since $R(E_x(u)) \subseteq F \setminus F'$ the existence of $E_x(u)$, it just only makes paths not cycles. By Lemma 3, for each vertex $v \in F''$, $R(v) \subseteq F \setminus F''$. Thus, the existence of F' and F'' would not make any cycle in $\overrightarrow{S_n}$. By Lemma 1, F is a feedback vertex set of $V(\overrightarrow{S_n})$ with cardinality $n!/2$. So $\mu(\overrightarrow{S_n}) \le n!/2 - 2(n-2)!$. □

In addition to give the upper bound to the minimum cardinality of the feedback vertex set for the n-dimensional directed spilt-star, we also give the lower bound to the minimum cardinality of the feedback vertex set for the n-dimensional directed spilt-star.

Theorem 5. $n \ge 4$, $\mu(\overrightarrow{S_n}) \ge n!/3$

Proof. In $\overrightarrow{S_4}$, there exists 8 disjoint 3-cycles. In order to break cycles in $\overrightarrow{S_4}$, we have to delete at least 8 vertices, a vertex for each disjoint cycle. The labels of deleted vertices are in the following: [3214], [3241], [3142], [3124], [4123], [4132], [4213] and [4231]. For $|V(\overrightarrow{S_4})| = 24$ and $\mu(\overrightarrow{S_4}) = 8/24 = 1/3$. Again, since there are $n!\,/4!$ copies of $\overrightarrow{S_4}$ in $\overrightarrow{S_n}$ for $n \ge 4$, and in each copy, we need to delete at least eight vertices. Then results $\mu(\overrightarrow{S_n}) \ge (n!/4!) \times 8 = n!/3$. □

3.The Feedback Vertex Set of Directed Alternating Group Graphs

The *n-dimensional directed alternating group graphs* $\overrightarrow{A_n^2}$ is a directed graph, which is induced by the set of even (resp. odd) permutations of $\overrightarrow{S_n}$. It is a regular graph with degree 2(n−2). Since $\overrightarrow{S_n^2,_E}$ is isomorphic to $\overrightarrow{S_n^2,_O}$ via a 2-exchange, without loss of generality, we let $\overrightarrow{A_n}$ be the even permutation of $\overrightarrow{S_n}$. The cardinality of vertex set and edge set of $\overrightarrow{A_n}$ is n! /2 and (n−2)n!/2. Alternating group graphs have a highly recursive structure. $\overrightarrow{A_n}$ is made up of $n\,\overrightarrow{A_{n-1}}$. Figure 3 and 4 depict examples of $\overrightarrow{A_3}$ and $\overrightarrow{A_4}$, respectively.

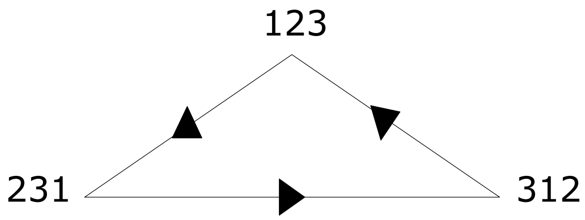

Figure 3: $\overrightarrow{A_3}$

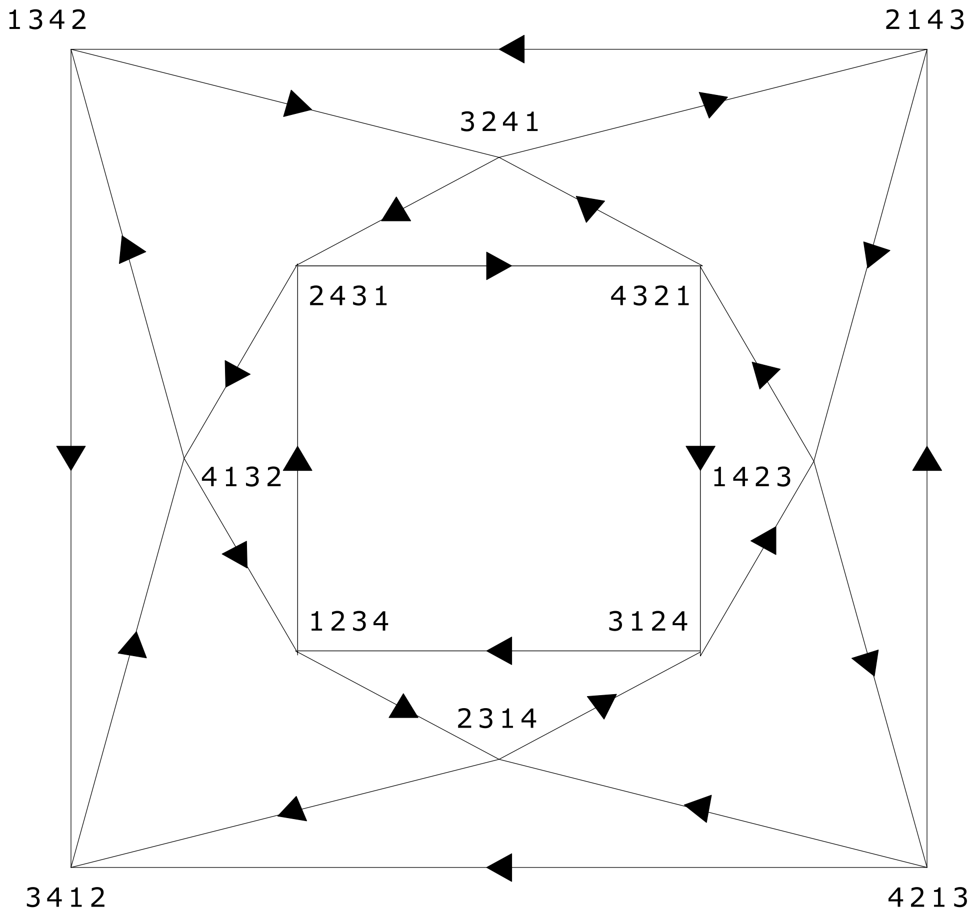

Figure 4: $\overrightarrow{A_4}$

Lemma 6. *Let* $F = \{[p_1,p_2,p_3,\ldots,p_n] \mid p_1 > p_2\}$ *and* $F \subseteq V(\overrightarrow{A_n})$*. Then F is a feedback vertex set of* $\overrightarrow{A_n}$.

Proof. Let F be a subset of $V(\overrightarrow{A_n})$ with cardinality $n!/4$. We want to show that F is a feedback vertex set of $\overrightarrow{A_n}$. Suppose, to the contrary, that F is not a feedback vertex set of $\overrightarrow{A_n}$. Then a cycle $C = u_1 \to u_2 \to \ldots \to u_k \to u_1$ exists in $\overrightarrow{A_n} \setminus F$. For each vertex $u_i = [u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,n}] \in C$. Since $u_{i,1} < u_{i,2}$, u_i has no 2-exchange neighbor. Therefore, u_{i+1} is a 3-rotation neighbor of u_i, $1 \le i \le k-1$, and u_1 is u_k's 3-rotation neighbor. Further, $u_{i,1} = u_{i+1,2}$ and $u_{k,1} = u_{1,2}$. Since $u_{i+1,1} < u_{i+1,2}$, $u_{i+1,1} < u_{i,1}$. Thus, $u_{k,1} < u_{k-1,1} < \ldots < u_{11} < u_{k,1}$, which is a contradiction. □

Lemma 7. *Let* $A' \subseteq V(\overrightarrow{A_n})$ *and* $A' = \{[2,1,p_3,\ldots,p_n] \mid p_i \in N_{1,2}$, for each $3 \le i \le n\}$, *for each vertex* $u \in A'$, $R(u) \subseteq F \setminus A'$.

Proof. Let $\overrightarrow{uw} \in \delta(u)$, $w \in R(u)$. If $w_i \in R(u)$, it has the form $[p_i, 2, p_3, \ldots p_{i-1}, 1, p_{i+1}, \ldots, p_n]$ and $p_i \ge 2$, $i \in N_{1,2}$. Since $R(u) \in F$ and $R(u) \notin A'$, $R(u) \subseteq F \setminus A'$. □

Lemma 8. *Let* $A'' \subseteq V(\overrightarrow{A_n})$ *and* $A'' = \{[n, n-1, p_3, \ldots, p_n] \mid p_i \in N_{n,n-1}$, for each $3 \le i \le n\}$, *for each vertex* $v \in A''$, $R(v) \subseteq F \setminus A''$.

Proof. Let $\overrightarrow{sv} \in \rho(v)$, then $s \in R(v)$. If $s_i \in R(v)$, it has the form $[n-1, p_i, p_3, \ldots p_{i-1}, n, p_{i+1}, \ldots, p_n]$ and $p_i \le n-1$, $i \in N_{1,2}$. Since $R(v) \in F$ and $R(v) \notin A''$, $R(v) \subseteq F \setminus A''$. □

Based on the Lemmas 6, 7 and 8, we give the following algorithm for solving the feedback vertex set problem in the directed alternating group graphs.

Algorithm FDA

Input: A directed alternating group graphs $\overrightarrow{A_n}$.

Output: A feedback vertex set of $\overrightarrow{A_n}$.

Method:

Step 1: $F = \{[p_1, p_2, p_3 \ldots p_n] \mid p_1 > p_2\}$

$F' = \{[2,1,p_3,\ldots,p_n] \mid p_i \in N_{1,2}$, for each $3 \le i \le n\}$

$F'' = \{[n, n-1, p_3, \ldots, p_n] \mid p_i \in N_{n,n-1}$, for each $3 \le i \le n\}$

Step 2: $S = A / (A' \cup A'')$.

Step 3: **output** S.

Lemma 6, Lemma 7 and Lemma 8 can derive the upper bound to the minimum cardinality of the feedback vertex set for the n-dimensional directed alternating group graphs.

Theorem 9. $\mu(\overrightarrow{A_n}) \leq n!/4-(n-2)!$

Proof. For each vertex $u \in A'$, $v \in A''$, by Lemma 7, $R(u) \subseteq F \setminus A'$, $R(v) \subseteq F \setminus A''$. Thus, the existence of A' and A'' would not make any cycle in $\overrightarrow{A_n}$. By Lemma 6, F is a feedback vertex set of $V(\overrightarrow{A_n})$ with cardinality n! / 4. So $\mu(\overrightarrow{A_n}) \leq n!/4-(n-2)!$.

□

We also give the lower bound to the minimum cardinality of the feedback vertex set for the n-dimensional directed alternating group graphs.

Theorem 10. $n \geq 4$, $\mu(\overrightarrow{A_n}) \geq n!/6$

Proof. In $\overrightarrow{A_4}$, there exists 4 disjoint 3-cycles. To break all cycles of $\overrightarrow{A_4}$, we need to prune at least 4 vertices, a vertex for each disjoint cycle. The labels of deleted vertices are in the following: [3241], [3124], [4132] and [4213]. For $|V(\overrightarrow{A_4})| = n!/2 = 12$ and $\mu(\overrightarrow{A_n}) = 4/12 = 1/3$. Again, since there are $n!/24$ copies of $\overrightarrow{A_4}$ in $\overrightarrow{A_n}$ for $n \geq 4$, and in each copy, we need to delete at least four vertices. Then results $\mu(\overrightarrow{A_n}) \geq (n!/24) \times 4 = n!/6$.

□

4. The Feedback Vertex Set of Undirected Spilt-stars

An *independent set* in a graph G is a vertex set $I \subseteq V(G)$ that contains no edge of G, that is to say $G[I]$ has no edge. Let $N', N'' \subseteq N$, where

$N' = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and $N'' = \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n\}$.

Lemma 11. *Let* $I = \{[x, y, p_3, \ldots, p_n] \mid x \in N', y \in N'', p_i \in N_{x,y}$ *and* $i = 3, 4, \ldots n\}$ *and vertex set* I *is an maximal independent vertex set of* S_n.

Proof. Since $I \subseteq V(S_n)$, we immediately show that for any two vertices u, $v \in I$, vertices u, v are not adjacent. Suppose, to the contrary, that I is not an independent vertex set of S_n. Then an edge $\overline{uv}$ exists in $G[I]$. Let

$u=\{[u_1,u_2,u_3,\ldots,u_n] \mid u_1\in N', u_2\in N''$ and $u_i\in N_{u1,u2}$ and $i=3,4,\ldots n\}$ and

$v=\{[v_1,v_2,v_3,\ldots,v_n] \mid v_1\in N', v_2\in N'', v_i\in N_{v1,v2}$ and $i=3,4,\ldots n\}$. Hence, v is either the 2-exchange neighbor of u or a 3-rotation neighbor of u. If v is the 2-exchange neighbor of u, then $v_1=u_2$, and $v_2=u_1$. Since $v_1\in N'$, $u_2\in N''$ and $N'\cap N''=\phi$, it is a contradiction. Otherwise, v is a 3-rotation neighbor of u. Thus, $v_2=u_1$or $v_1=u_2$. Similarly, it contradicts that $N'\cap N''=\phi$.

Furthermore, we shall prove that I is maximal. Suppose, to the contrary, that I is not a maximal independent set of S_n. Then, there exists a vertex $v\in V(S_n)\setminus I$ and $I\cup\{v\}$ is also a independent set of S_n . That is to say, u and v are nonadjacent, for each $u\in I$. Let $u=\{[u_1,u_2,u_3,\ldots,u_n] \mid u_1\in N', u_2\in N''$and $u_i\in N_{u1,u2}$ and $i=3,4,\ldots n\}$. Since $v\notin I$, v belongs to one of the following three vertex sets.

(1) $V'=\{[v_1,v_2,v_3\ldots,v_n] \mid v_1\in N'$ and $v_2\in N', v_i\in N_{v1,v2}$ and $i=3,4,\ldots n\}$,

(2) $V'''=\{[v_1,v_2,v_3\ldots,v_n] \mid v_1\in N''$ and $v_2\in N'', v_i\in N_{v1,v2}$ and $i=3,4,\ldots n\}$,

(3) $V''=\{[v_1,v_2,v_3\ldots,v_n] \mid v_1\in N''$ and $v_2\in N', v_i\in N_{v1,v2}$ and $i=3,4,\ldots n\}$.

Now, we discuss it according to the listed classes.

Case 1: $v\in V'$. Let $w=[v_2,v_i,\ldots,v_{i-1},v_1,v_{i+1},\ldots,v_n]\in N(v)$, where $v_i\in N''$. Then $w\in I$, $\overline{vw}\in E(S_n)$. Which contradicts that v, w are nonadjacent, for each vertex $w\in I$.

Case 2: $v\in V'''$. The proof is similar to case 1.

Case 3: $v\in V''$. Let $w=[v_2,v_1,\ldots,v_{i-1},v_i,v_{i+1},\ldots,v_n]\in N(v)$, where $v_2\in N'$and $v_1\in N''$. Then $w\in I$, $\overline{vw}\in E(S_n)$. It is a contradiction. □

According to the size of N' and N'', we get the following result $|I|=(n^2/4)(n-2)!$, if n is even and $|I|=(n^2-1/4)(n-2)!$, if n is odd.

Lemma 12. Let $L'=\{[x,y,p_3\ldots,p_n] \mid x,y\in N', x=1,3,5,\ldots,\lfloor n/2\rfloor$, $y=x+1, p_i\in N_{x,y}$ *and* $i=3,4,\ldots$n$\}$, *then* L' *is an independent set of* S_n.

Proof. For each vertex u, $v\in L'$, let $u=[x_1, y_1, p_3\ldots p_n]$ and $v=[x_2, y_2, p_3\ldots, p_n]$. Suppose, to the contrary, that L' is not an independent vertex set of S_n. Then an edge $\overline{uv}$ exists in $G[L']$. Hence, v is either the 2-exchange neighbor of u or a 3-rotation neighbor of u. If v is the 2-exchange neighbor of u, then $x_1 = y_2$, and $y_1 = x_2$. Since x_1, x_2 are odd and y_1, y_2 are even, it is a contradiction. Otherwise, v is a 3-rotation neighbor of u. Thus, $y_2 = x_1$ or $x_2 = y_1$. Similarly, it contradicts that x_1, x_2 are odd and y_1, y_2 are even. Therefore, L' is an independent set of S_n. □

Lemma 13. *Let* $L'' = \{u \mid v \in L', u \in E_x(v)\}$, *then* L'' *is an independent set of* S_n.

Proof. Since L'' is isomorphism to L', thus L'' is an independent set of S_n. An *independent edge set* in a graph G is an edge set $E' \subseteq E(G)$ that each edge contains no common vertex of G, that is to say $G[E']$ has no cycle.

Lemma 14. *Let* $L'' = \{u \mid v \in L', u \in E_x(v)\}$, *then* $G(L' \cup L'')$ *is an independent edge set in* S_n.

Proof. For each vertex $u, v \in L'$, $s, w \in L''$. s is a 2-exchange neighbor of u and w is a 2-exchange neighbor of v .We assume that $\overline{us}$ and $\overline{vw}$ have a common vertex . Let $u = [x_1, y_1, p_3\ldots, p_n]$, $s = [y_1, x_1, p_3\ldots, p_n]$, $v = [x_2, y_2, p_3\ldots, p_n]$ and $w = [y_2, x_2, p_3\ldots, p_n]$. Without loss generality let $s = v$ be the common vertex of $\overline{us}$ and $\overline{vw}$. Then, $y_1 = x_2$ and $x_1 = y_2$. Since u, $v \in L'$ and $x_1 \neq y_1 \neq x_2 \neq y_2$. It is a contradiction. □

A star is $K_{1,n}$ for some $n \geq 2$. A *doubled-rooted star* (*DRS*) is the union of 2 $K_{1,n}$, plus an edge between 2 vertices with maximum degrees. For example, Figure 5 (*a*) there are two $K_{1,5}$ and Figure 5 (*b*) is double-rooted star constructed from 5 (*a*) with one more edge.

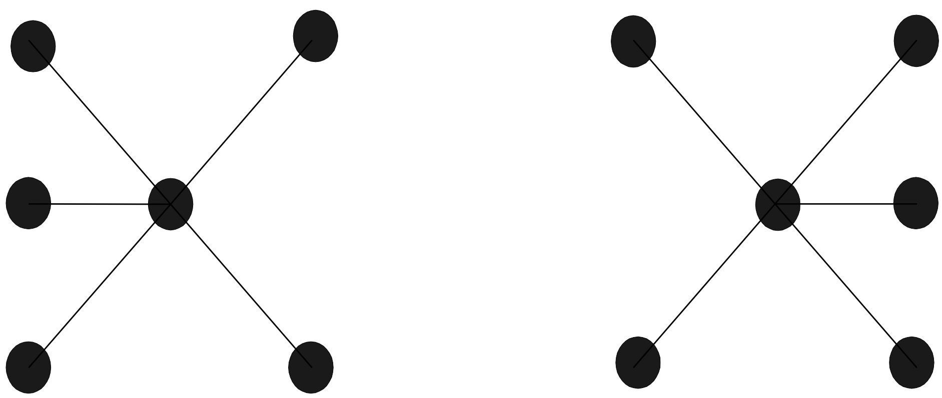

Figure 5 (*a*)

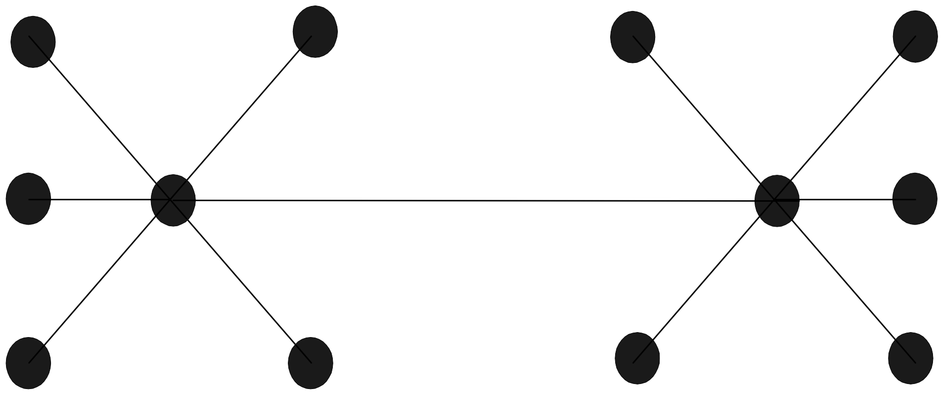

Figure 5 (b)

Lemma 15. $G(L' \cup L'' \cup I)$ is acyclic and $G(L' \cup L'' \cup I)$is a union of disjoint double rooted stars.

Proof. Let $u=[p_1, p_2, p_3\ldots, p_n]\in I$. If p_1 is odd (even), then there exists a unique $p_i \in N'$ and $p_i=p_1+1$ ($p_i=p_1-1$) such that $v=[p_i, p_1, p_3,..,p_{i-1}, p_2\ p_{i+1},..,p_n]\in L''(L')$, respectively. Then, v and its 2-exchange neighbor are the roots of a double-rooted star. Since each $u\in I$ is uniquely connected to a double-rooted star, $G(L' \cup L'' \cup I)$ is a union of disjoint double rooted stars. Thus, $G(L' \cup L'' \cup I)$ is acyclic. □

Lemma 16. *Let* $L''' = \{[n, n-1, p_3\ldots, p_n] \mid [n, n-1, p_3\ldots, p_n] \cap A_n, p_i\in N_{n,n-1}$, *and* $i=3,4,\ldots n\}$, *and* $I'\subseteq I$, $I'=\{[p_1, n, p_3\ldots,p_n] \mid p_1\in N', p_i\in N_{p1,n}$ *and* $i=3,4,\ldots n\}$, *then* $N(L''')\cap I \subseteq I'$.

Proof. Let $N(L''')\cap I=\Gamma_1\cup\Gamma_2$, where

$\Gamma_1=\{[n-1, p_i, p_3,\ldots p_{i-1}, n, p_{i+1}\ldots, p_n] \mid p_i\in N_{n-1,n}, i=3,4,\ldots n\}$ and

$\Gamma_2=\{[p_i, n, p_3,\ldots p_{i-1}, n-1, p_{i+1}\ldots, p_n] \mid p_i\in N_{n-1,n}, i=3,4,\ldots n\}$.

Therefore, $N(L''')\cap I=(\Gamma_1\cap I)\cup(\Gamma_2\cap I)$. Now, we want to compute $\Gamma_i\cap I$, $i=1,2$, to complete the proof. Since $n-1\notin N'$, $(\Gamma_1\cap I)=\phi$. To find $\Gamma_2\cap I$, $\Gamma_2\cap I\subseteq I'$ because $p_i\in N'$ and $n\in N''$. Thus $N(L''')\cap I\subseteq I'$. □

Lemma 17. *For any two distinct vertex a,* $b\in L'''$, *let* $N_{I'}(a)=N(a)\cap I'$, $N_{I'}(b)=N(b)\cap I'$, *then* $N_{I'}(a)\cap N_{I'}(b)=\phi$.

Proof. Let $a=[n, n-1, a_3,\ldots,a_n]$ and $b=[n, n-1, b_3,\ldots,b_n]$.Since

$N_{I'}(a)$, $N_{I'}(b) \subseteq I'$, therefore $N_{I'}(a) = [a_i, n, a_3,.. a_{i-1}, n-1, a_{i+1}\ldots, a_n]$, $a_i \in N'$ and $N_{I'}(b) = [b_j, n, b_3,\ldots b_{j-1}, n-1, b_{j+1}\ldots, b_n]$, $b_j \in N'$. Let for any s be the common neighbor of a and b, then $s = [p_k, n, p_3,\ldots p_{k-1}, n-1, p_{k+1}\ldots, p_n]$, $p_i \in N'$. For such the position of $n-1$ that $i=j=k$ and $a_i = p_k = b_j$, $a_{i-1} = p_{k-1} = b_{j-1}$, $a_{i+1} = p_{k+1} = b_{j+1}$. It implies that $a = b$, but it contradicts to

$a \neq b$, thus $N_{I'}(a) \cap N_{I'}(b) = \phi$. □

Lemma 18. *$G(L' \cup L'' \cup L''')$ contains no cycle in S_n.*

Proof. Let $v = [n, n-1, p_3,\ldots, p_n] \in L'''$. For each $u \in N(v)$, there are three possible forms of u in the following.

Case(1): $u = [n-1, n, p_3,\ldots, p_n]$. Since $n-1$ and $n \notin N'$, $u \notin L' \cup L''$.

Case(2): $u = [n-1, p_i, p_3,\ldots p_{i-1}, n, p_{i+1}\ldots, p_n]$. Since $n-1 \notin N'$, $u \notin L' \cup L''$.

Case(3): $u = [p_i, n, p_3,\ldots p_{i-1}, n-1, p_{i+1}\ldots, p_n]$. Since $n \notin N'$, $u \notin L' \cup L''$. □

Lemma 19. *If $u \in L'''$ then u can connect to at most one vertex v in each double rooted star of* $G(L' \cup L'' \cup I')$.

Proof We define Π_i, $i = 1,3,5,\ldots \lfloor n/2 \rfloor$, denote the set of double rooted star which roots are labeled with $[i, i+1, p_3,\ldots, p_n]$ and $[i+1, i, p_3,\ldots, p_n]$. Let $u = [n, n-1, p_3,\ldots, p_n] \in L'''$. For each $v \in N(u)$, there are three possible forms of v in the following.

Case(1): $v = [n-1, n, p_3,\ldots, p_n]$. Since $n-1 \notin N'$, $v \notin L' \cup L'' \cup I'$. That is to say u does not adjacent with any vertex of $G(L' \cup L'' \cup I')$.

Case(2): $v = [n-1, p_i, p_3,\ldots p_{i-1}, n, p_{i+1}\ldots, p_n]$. This proof is similar to

case (1).

Case(3): $v = [p_i, n, p_3,\ldots p_{i-1}, n-1, p_{i+1}\ldots, p_n]$. If v is in no *DRS*, then u does not adjacent with any *DRS*. $v \in \Pi_{pi}$ if p_i is odd and $v \in \Pi_{pi-1}$, otherwise. For each Π_i, since the second symbol of each roots is less than n, v is not a root. By the construction of Π_i, there are exactly two leaves $v_1 = [i, n, x_3, x_4,\ldots x_n]$ and $v_2 = [i+1, n, x_3, x_4,\ldots x_n]$ with the permutation that the second symbol is n. since $v_1 \neq v_2$, either v_1 or v_2 is the only neighbor of u.

□

Theorem 20. *$G(L' \cup L'' \cup I \cup L''')$ is acyclic.*

Proof. By lemma 15 $G(L' \cup L'' \cup I)$ is acyclic and by Lemma 19, u can connect to at most one vertex in each *DRS*. Thus, there is no cycles in $G(L' \cup L'' \cup I \cup L''')$. □

Based on the Theorem 20, we give the following algorithm for solving the feedback vertex set problem in the undirected split-stars.

Algorithm FUS

Input: An undirected split-star S_n.

Output: A feedback vertex set of S_n.

Method:

Step 1: $I = \{[x,y, p_3,\ldots,p_n] \mid x \in N', y \in N'', p_i \in N_{x,y} \text{ and } i = 3,4,\ldots n\}$.

$L' = \{[x,y, p_3\ldots,p_n] \mid x, y \in N', x = 1,3,5,\ldots,\lfloor n/2 \rfloor, y = x+1, p_i \in N_{x,y} \text{ and } i = 3,4,\ldots n\}$.

$L'' = \{u \mid v \in L', u \in E_x(v)\}$.

$L''' = \{[n, n-1, p_3\ldots, p_n] \mid [n, n-1, p_3\ldots, p_n] \cap A_n, p_i \in N_{n,n-1}, \text{ and } i = 3,4,\ldots n\}$.

Step 2: $S = I \cup L' \cup L'' \cup L'''$.

Step 3: **output** S.

Since $G(L' \cup L'' \cup I \cup L''')$ is acyclic, $G(L' \cup L'' \cup I \cup L''')$ is a feedback vertex set, we immediately have the following result.

Theorem 21. $\mu(S_n) \le n! - [(n^2+2n/4)(n-2)! + (n-2)!/2]$, if n is even.

$\mu(S_n) \le n! - [(n^2-1+2n/4)(n-2)! + (n-2)!/2]$, if n is odd.

Any connected acyclic graph must be a tree. To determine a given simple graph G is acyclic or not, we make use of the relationship between number of vertices and edges in each component of G. Furthermore, the following lemma applied to find the lower bound of the undirected spilt-stars.

Lemma 22. *Let G be a simple graph. G is cyclic, if* $|V(G)| \le |E(G)|$.

Proof. Without loss of generality, we may assume G is connected. Otherwise there is a cycle in a small component (by induction). If G does not contains a cycle, the G is a tree then $|E(G)| = |V(G)| - 1$. □

An edge is called *outer-edge* if the endvertices of this edge belong to two different substars and the cardinality of outer-edges of some vertex v is the *outer-degree* of v. Otherwise, an edge is called *inner-edge* if the endvertices of this edge belong to the same substars and the cardinality of inner-edges of some vertex v is the *inner-degree* of v. For example, Figure 2 shows the 24 outer-edges in S_4. Let we denote the degree of v in graph G by $deg_G(v)$.

Lemma 23. $\mu(S_4) \geq 11$.

Proof. The 4-dimensional split-star graph S_4 can be recursively constructed by four 3-dimensional split-star as its subgraph, named S_3, and each S_3 contains two vertex disjoint 3-cycles. To count the cardinality of the 3-cycles in S_4, it can be seen that since each vertex in S_4 incidents with two 3-cycles and each 3-cycle is repeatedly counted three times, there are totally sixteen 3-cycles in S_4. For each vertex disjoint 3-cycle, we must delete at least one vertex to ensure the acyclic, and so this vertex results in two 3-cycles be broken. Then each vertex in S_3 is forced to loss its degree by 2, for the removal of the two cycles, which the vertex belongs. Therefore, each vertex has one or two inner-degree less than the original vertex, because they have to adjacent to at least one of the two vertices deleted. We find the out-degree of each vertex in S_4 is two. Since each vertex incidents with two 3-cycles and if we delete eight vertices for each vertex disjoint 3-cycle then we discredit sixteen 3-cycles. So the out-degree less than or equals to one for each vertex u in the remaining graph. It is clear $deg(u) \leq 2 + 1 = 3$ and $\mu(S_4) \geq 8$. After we prune eight vertices in S_4, there are at least 20 edges and 16 vertices left. By Lemma 22, there still exists cycles in the remaining graph. Then, we further delete vertices to let the remaining graph to be acyclic. For each vertex u with degree 3, since u is adjacent with at most one vertex in S_3 with degree 3, there are at most two neighbors of u with degree 3. So, we first remove one of the eight vertices, v_1, with degree 3, and then there are at least five vertices with degree three exist in the remaining graph. Furthermore, we delete another vertex v_2 with degree 3. There remain two vertices with degree three. Again, we can cancel one of these two vertices to break all the cycles of the remaining graph. Thus, the remaining graph is acyclic and $\mu(S_4) \geq 11$. □

Theorem 24. $n \geq 4$, $\mu(S_n) \geq (11/24)n!$

Proof In order to break cycles in S_4, we have to delete at least 11 vertices. The labels of deleted vertices are in the following: [3124], [3142], [4123], [4132], [3214], [3241], [4213], [4231], [3412], [3421] and [4312]. For $|V(S_4)| = 24$ and $\mu(S_4) = 11$. Again, since there are $n!$ /4! copies of S_4 in S_n for $n \geq 4$, and in each copy, we need to delete at least eleven vertices. Then results $\mu(S_n) \geq (n!/4!)\times 11 = (11/24)n!$. □

5. The Feedback Vertex Set of undirected Alternating Group Graphs

The constructions of undirected alternating group graphs are the same as directed alternating group graphs except that the direction of every edge is bi-directional. For simplicity, we discard the arc's directions of undirected alternating group graphs. Figure 6 depicts example of A_4 .

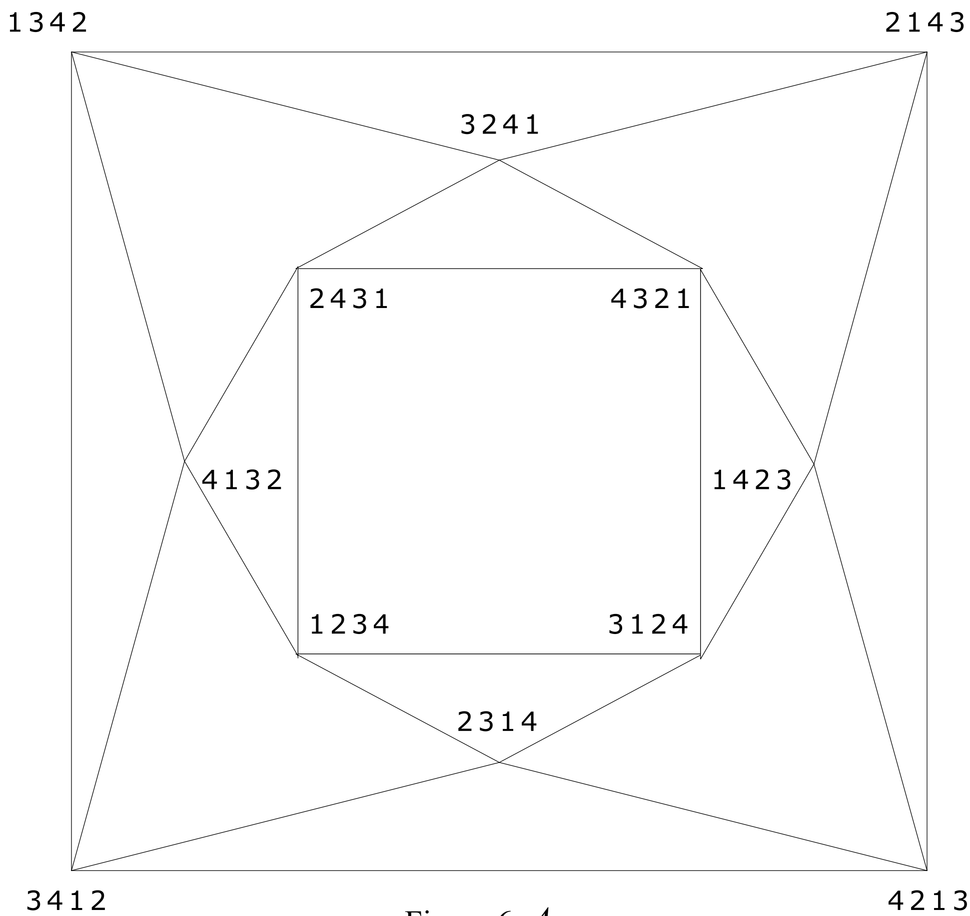

Figure 6: A_4

Here, we also use the result of independent set of spilt-star to implement the alternating group graphs.

Lemma 25. *Let* $I = \{[a,b, p_3 ... p_n] \mid a \in N', by \in N'', p_i \in N_{a,b}$ *and*

$i=3,4,\ldots n\}$ and vertex set I is an maximal independent vertex set of A_n.

Proof. Since $I \subseteq V(A_n)$, we immediately show that for any two vertices u, $v \in I$, vertices u, v are not adjacent. Suppose, to the contrary, that I is not an independent vertex set of S_n. Then an edge $\overline{uv}$ exists in $G[I]$. Let

$u=\{[u_1,u_2,u_3\ldots u_n] \mid u_1 \in N', u_2 \in N''$ and $u_i \in N_{u1u2}$ and $i=3,4,\ldots n\}$ and

$v=\{[v_1,v_2,v_3\ldots v_n] \mid v_1 \in N', v_2 \in N'', v_i \in N_{v1v2}$ and $i=3,4,\ldots n\}$. Hence, v is either the 2-exchange neighbor of u or a 3-rotation neighbor of u. If v is the 2-exchange neighbor of u, then $v_1=u_2$, and $v_2=u_1$. Since $v_1 \in N'$,

$u_2 \in N''$ and $N' \cap N'' = \phi$, it is a contradiction. Otherwise, v is a 3-rotation neighbor of u. Thus, $v_2=u_1$or $v_1=u_2$. Similarly, it contradicts that $N' \cap N'' = \phi$.

Furthermore, we shall prove that I is maximal. Suppose, to the contrary, that I is not a maximal independent set of A_n. Then, there exists a vertex $v \in V(A_n) \setminus I$ and $I \cup \{v\}$ is also a independent set of A_n . That is to say, u and v are nonadjacent, for each $u \in I$. Let $u=\{[u_1,u_2,u_3\ldots u_n] \mid u_1 \in N', u_2 \in N''$ and $u_i \in N_{u1,u2}$ and $i=3,4,\ldots n\}$. Since $v \notin I$, v belongs to one of the following three vertex sets.

(1) $V' = \{[v_1,v_2,v_3\ldots v_n] \mid v_1 \in N'$ and $v_2 \in N', v_i \in N_{v1,v2}$ and $i=3,4,\ldots n\}$,

(2) $V''' = \{[v_1,v_2,v_3\ldots v_n] \mid v_1 \in N''$ and $v_2 \in N'', v_i \in N_{v1,v2}$ and $i=3,4,\ldots n\}$,

(3) $V'' = \{[v_1,v_2,v_3\ldots v_n] \mid v_1 \in N''$ and $v_2 \in N', v_i \in N_{v1,v2}$ and $i=3,4,\ldots n\}$.

Now, we discuss it according to the listed classes.

Case 1: $v \in V'$. Let $w=[v_2,v_i,\ldots,v_{i-1},v_1,v_{i+1},\ldots,v_n] \in N(v)$, where $v_i \in N''$. Then $w \in I$, $\overline{vw} \in E(S_n)$. Which contradicts that v, w are nonadjacent, for each vertex $w \in I$.

Case 2: $v \in V'''$. The proof is similar to case 1.

Case 3: $v \in V''$. Let $w=[v_2,v_1,\ldots,v_{i-1},v_i,v_{i+1},\ldots,v_n] \in N(v)$, where $v_2 \in N'$ and $v_1 \in N''$. Then $w \in I$, $\overline{vw} \in E(A_n)$. It is a contradiction. □

Based on the theorem 20 and Lemma 25, we give the following algorithm for solving the feedback vertex set problem in the undirected alternating group graphs.

Algorithm FUA

Input: An undirected alternating group graphs A_n.

Output: A feedback vertex set of A_n.

Method:

Step 1: $I=\{[x,y,p_3,\ldots,p_n] \mid x\in N', y\in N'', p_i\in N_{x,y}$ *and* $i=3,4,\ldots n\}$.

$L'=\{[x,y,p_3\ldots,p_n] \mid x, y\in N', x=1,3,5,\ldots,\lfloor n/2\rfloor, y=x+1, p_i\in N_{x,y}$ *and* $i=3,4,\ldots n\}$.

$L''=\{u \mid v\in L', u\in E_x(v)\}$.

$L'''=\{[n,n-1,p_3\ldots,p_n] \mid [n,n-1,p_3\ldots,p_n] \cap A_n, p_i\in N_{n,n-1}$, *and* $i=3,4,\ldots n\}$.

Step 2: $S=I\cup L'\cup L''\cup L'''$.

Step 3: **output** S.

$|I|=(n^2/4)(n-2)!/2$, if n is even and $|I|=(n^2-1/4)(n-2)!/2$, if n is odd.

By Theorem 20, we have shown that $G(L'\cup L''\cup I\cup L''')$ is acyclic. Since $G(L'\cup L''\cup I\cup L''')$ is acyclic, $G(L'\cup L''\cup I\cup L''')$ is a feedback vertex set, we immediately have the following result.

Theorem 26. $\mu(A_n)\le n!/2-[(n^2/4)(n-2)!/2+(n-2)!/4]$, if n is even.

$\mu(A_n)\le n!/2-[(n^2-1/4)(n-2)!/2+(n-2)!/4]$, if n is odd.

We also make use of the relationship between number of vertices and edges in each component of A_4 to find the lower bounds of the undirected alternating group graphs.

To decide the lower bound of the feedback vertex number, By Lemma 22 and Lemma23, an important observation is established as follows. In A_4, the graph is exactly covered by 4 disjoint vertex 3-cycles. To break all cycles of A_4, we discard four vertices, a vertex for each disjoint cycle, at first. And cycle does not survive in any A_3. It is clear that $\mu(A_4)\ge 4$, and since there are 24 edges in A_4, it is easy to see that there are at least 8 edges left. To cut the remaining cycles, one edge should be pruned at least, because there are 8 vertices survived in the remaining graph. Therefore, it is necessary to remove one vertex in the remaining graph to break all cycles of A_4. Thus, $\mu(A_4)\ge 5$ and the lower bound of A_n is built as follows.

Theorem 27. $n\ge 4, \mu(A_n)\ge(5/24)n!$

Proof To break cycles in A_4, we have to omit at least 5 vertices. The labels of deleted vertices are in the following: [3124], [4132], [3241], [4213] and [3412]. For $|V(A_4)|=12$

and $\mu(A_4) = 5$. Again, since there are $n!\,/4!$ copies of A_4 in A_n for $n \geq 4$, and in each copy, we need to delete at least five vertices. Then results $\mu(A_n) \geq (n!/4!) \times 5 = (5/24)n!$. □

6. Concluding Remarks

A recent line of research on polynomially solvable cases focuses on special undirected graphs having bounded degree and that are widely used as connection networks, namely meshes and toroidal meshes,

Butterflies, toroidal butterflies, and hypercubes. In meshes and toroidal meshes, Luccio [10] obtained the upper bounds on the size of the minimum feedback vertex set. These bounds either match the lower bounds or are very close to them. For butterfly graphs, Luccio [10] found both bounds to the size of a minimum feedback vertex set. Similar results to those obtained for butterflies can also be obtained for toroidal butterflies.

Spilt-stars, an alternative to the star graphs, are companion graphs to alternating group graphs. These graphs have many advantages over the n-cubes. Recently, Cheng et al. [1] proposed an orientation to the spilt-stars and the alternating group graphs. They showed that the oriented graphs are maximally arc-connected and have small diameters. In this thesis, we study the feedback vertex set problem on directed and undirected spilt-stars and alternating group graphs separately. At the first part, the upper and lower bounds to the feedback vertex set for the directed spilt-stars and alternating group graphs, respectively, are determined. At the second part, we give the both bounds to the undirected spilt-stars and alternating group graphs by expanding maximal independent sets, respectively, to decide the feedback vertex sets.

In the construction of the remaining graph, discard the feedback vertex set from the given undirected graph, we add a specified maximal independent set with undirected other vertices. However, the independent set we used is not maximum. Further, a natural question to ask a maximum independent set to increase the size of feedback vertex set is our next research. And we can also study the feedback vertex set for the other topologies such as multi-mesh or star graph.

7. Reference

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