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Fault-tolerant Hamiltonian laceability of bipartite hypercube-like networks

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Abstract*-The fault tolerance for Hamiltonian properties of n-dimensional bipartite hypercube-like graphs X_n are explored. The main result of this paper is that $X_n - F$ is Hamiltonian laceable where F is the faulty edge set with $|F| = n - 3$.*

Keywords: hypercube-like, Hamiltonian laceable, fault tolerance.

1. Introduction[1]

The hypercube network is one of the most popular interconnection networks. It has many attractive properties, such as regularity, symmetry, low diameter, simple routing algorithms[2]. The hypercube network also contains some Hamiltonian properties[3].

Let $G = (V_0 \cup V_1, E)$ be a *bipartite* graph where V_0 and V_1 are two disjoint vertex sets such that each edge of E consists of one vertex from each set. Two vertices a and b are *adjacent* if $(a, b) \in E$. A *path* is a sequence of adjacent vertices, denoted as $\langle v_1, v_2, \ldots, v_n \rangle$, where all the vertices $v_1, v_2, \ldots, v_n$ are distinct. A *Hamiltonian path* is a path that spans G. A *cycle*, written as $\langle v_1, v_2, \ldots, v_n \rangle$, is a path for $v_1 = v_n$. A *Hamiltonian cycle* is a cycle that traces every vertex exactly once. A *Hamiltonian graph* is a graph that contains a Hamiltonian cycle. A Hamiltonian graph G is *k edge Hamiltonian* if $G - F$ is Hamiltonian for every $F \subset E(G)$ with $|F| = k$. A bipartite graph $G = (V_0 \cup V_1, E)$ is *Hamiltonian laceable* if there is a Hamiltonian path between every pair of vertices x and y for $x \in V_0$ and $y \in V_1$. A *k edge Hamiltonian laceable graph G* is a Hamiltonian laceable graph that $G - F$ is Hamiltonian laceable for every $F \subset E(G)$ with $|F| = k$. A Hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is *hyper-Hamiltonian laceable* if $\forall v \in V_i$, $i = 0, 1$, there is a Hamiltonian path of $G - v$ between every pair of vertices of V_{1-i}. In [5], Tsai et al. showed that the hypercube Q_n is $(n-2)$ edge Hamiltonian laceable and $(n-3)$ edge hyper-Hamiltonian laceable.

In [6], Vaidya et al. introduced the class of hypercube-like graphs. The class of HL-graphs contains most of hypercube variants. Park et al. showed that the bipartite HL-graph is Hamiltonian laceable and the non-bipartite HL-graph is Hamiltonian connected[4]. In this paper, we show that the bipartite HL-graph X_n is $(n-3)$ edge Hamiltonian laceable and $(n-2)$ edge Hamiltonian.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs with $|V_1| = |V_2|$. The cross edge set $E = \{(v, \phi(v)) \mid v \in V_1, \phi(v) \in V_2$ and $\phi : V_1 \to V_2$ is a bijection$\}$. Let $G_1 \oplus G_2$ denote $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$. Let G be partitionable if $G = G_1 \oplus G_2$ for some graphs G_1 and G_2. For convenience, we denote the neighbor of vertex u as $\phi(u)$ for $(u, \phi(u)) \in E$. The n-dimensional HL-graphs HL_n can be defined as:

(1) $HL_0 = \{G_0\}$, where G_0 is a trivial graph (which has only one vertex).
(2) $G \in HL_{n+1}$ if and only if $G = G_1 \oplus G_2$ for some $G_1, G_2 \in HL_n$.

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In this paper, we focus on bipartite HL-graphs. Let X_n denote an arbitrary bipartite n-dimensional HL-graph. The two subgraphs of X_n denote with X_{n-1}^0 and X_{n-1}^1. Let F, F_0 and F_1 be the faulty edge sets of X_n, X_{n-1}^0 and X_{n-1}^1, respectively.

2. Hamiltonian laceability of the bipartite HL-graphs.

In this section, we will show the bipartite HL-graphs X_n is hyper-Hamiltonian laceable.

Theorem 1. *Every n-dimensional bipartite HL-graph is hyper-Hamiltonian laceable.*

Proof. The proof is by induction on n.

It is trivial for $n \leq 2$. By induction hypothesis, the two subgraphs X_{n-1}^0 and X_{n-1}^1 of X_n are hyper-Hamiltonian laceable. Let s and t be red nodes and w be a blue node. We will construct a Hamiltonian path of X_n - w between s and t in the following cases.

Case 1. *s, t and w are in the same subgraph.*

Without loss of generality, we can assume that s, t and w are in X_{n-1}^0. Since X_{n-1}^0 is hyper-Hamiltonian laceable, there exists a Hamiltonian path P_0 from s to t in $X_{n-1}^0 - w$. Let P_0 contain the edge (u, v) where vertex u is blue and vertex v is red. We can write the path P_0 as $\langle s \rightarrow P(s, u) \rightarrow u, v \rightarrow P(v, t) \rightarrow t \rangle$. Since X_{n-1}^1 is Hamiltonian-laceable, there exists a Hamiltonian path P_1 from $\phi(u)$ to $\phi(v)$. We can construct the Hamiltonian path from s to t as $\langle s \rightarrow P(s, u) \rightarrow u, \phi(u) \rightarrow P_1 \rightarrow \phi(v), v \rightarrow P(v, t) \rightarrow t \rangle$, as illustrated in Fig. 1(a).

Case 2. *s and t are in the same subgraph, w is in another subgraph.*

Without loss of generality, we can assume that s and t are in X_{n-1}^0, and w is in X_{n-1}^1. Let vertex p in X_{n-1}^0 be blue, and v be adjacent to p in X_{n-1}^0. Since X_{n-1}^0 is hyper-Hamiltonian laceable, there exists a Hamiltonian path P_0, denoted as $\langle s \rightarrow P(s, u) \rightarrow u, v \rightarrow P(v, t) \rightarrow t \rangle$, from s to t in $X_{n-1}^0 - p$ where (u, v) is an edge of P_0. Since X_{n-1}^1 is hyper-Hamiltonian laceable, there exists a Hamiltonian path P_1 from $\phi(u)$ to $\phi(p)$ in $X_{n-1}^1 - w$. We can construct the Hamiltonian path from s to t in $X_n - w$ *as* $\langle s \rightarrow P(s, u) \rightarrow u, \phi(u) \rightarrow P_1 \rightarrow \phi(p), p, v \rightarrow P(v, t) \rightarrow t \rangle$, as illustrated in Fig. 2(b).

Case 3. *s and t are in different subgraphs.*

Without loss of generality, we can assume that s and w are in X_{n-1}^0, and t is in X_{n-1}^1. Since X_{n-1}^0 is hyper-Hamiltonian laceable, there exists a Hamiltonian path P_0 denoted as $\langle s \rightarrow P(s, p) \rightarrow p \rangle$ from s to p in $X_{n-1}^0 - w$, where p is red. Since X_{n-1}^1 is Hamiltonian-laceable, there exists a Hamiltonian path P_1 from $\phi(p)$ to t. We can construct the Hamiltonian path from s to t in $X_n - w$ as $\langle s \rightarrow P_0 \rightarrow p, \phi(p) \rightarrow P_1 \rightarrow t \rangle$, as illustrated in Fig. 2(c).□

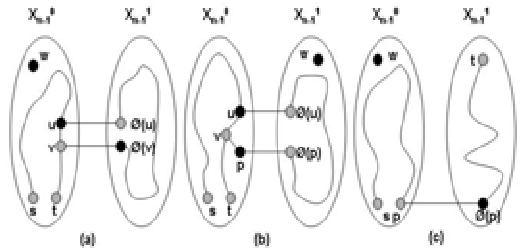

Fig. 1. (a) Case 1: s,t and w ∈ X_{n-1}^0. (b) Case 2: s, t ∈ X_{n-1}^0 and w ∈ X_{n-1}^1. (c) Case 3: s ∈ X_{n-1}^0 and t ∈ X_{n-1}^1

3. Fault-tolerant Hamiltonian laceability of the bipartite HL-graphs.

In this section, we will prove an n-dimensional bipartite HL-graph X_n is $(n$-3) edge Hamiltonian laceable and $(n$-2) edge Hamiltonian.

Lemma 1. The graph X_3 *is 1 edge Hamiltonian laceable.*

Proof. In [4], Tsai et al. show that the hypercube Q_3 is 1 edge Hamiltonian laceable. The bipartite HL-graph X_3 is isomorphic to Q_3. Thus X_3 is also 1 edge Hamiltonian laceable. □

Lemma 2. The graph X_4 is *1 edge Hamiltonian laceable.*

Proof. Let X_3^0 and X_3^1 be the two disjoint subgraphs of X_4. Let vertex s be red and vertex t

be blue.

Case 1. *s and t are in the same subgraph.*

Without loss of generality, we can assume that s and t are in X_3^0. Since X_3^0 is 1 edge Hamiltonian laceable, there exists a Hamiltonian path P_0 from s to t in X_3^0. Let (u, v) be an edge of P_0 such that the edges $(u, \phi(u))$ and $(v, \phi(v))$ are fault-free. We can write the path $P_0 = \langle x \rightarrow P(x, u) \rightarrow u, v \rightarrow P(v, t) \rightarrow t\rangle$. Since subgraph X_3^1 is 1 edge Hamiltonian laceable, there exists a Hamiltonian path P_1 from $\phi(u)$ to $\phi(v)$ in X_3^1. We can construct the Hamiltonian path from s to t in X_4 as $\langle s \rightarrow P(s, u) \rightarrow u, \phi(u) \rightarrow P_1 \rightarrow \phi(v), v \rightarrow P(v, t) \rightarrow t\rangle$, as illustrated Fig. 2(a).

Case 2. *s and t are in different subgraphs.*

Without loss of generality, we can assume that $s \in X_3^0$ and $t \in X_3^1$. Let p be a blue node in X_3^0 such that $(p, \phi(p))$ is fault-free. Since X_3^0 and X_3^1 are 1 edge Hamiltonian laceable, there exists a Hamiltonian path P_0 from s to p in X_3^0 and the Hamiltonian path P_1 from $\phi(p)$ to t in X_3^0 and X_3^1, receptivity. We can construct the Hamiltonian path from s to t in X_4 as $\langle s \rightarrow P_0 \rightarrow p, \phi(p) \rightarrow P_1 \rightarrow t\rangle$, as illustrated Fig. 2(b). □

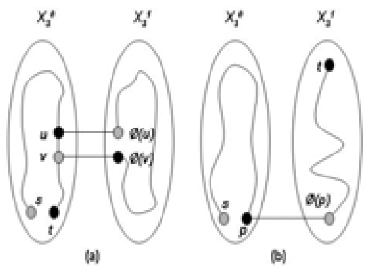

Fig. 2. (a) Case 1: $s, t \in X_3^0$ (b) Case 2: $s \in X_3^0$ and $t \in X_3^1$

In [4], Park et al. show the following lemma.

Lemma 3. For given two red vertices s, v and two blue vertices t, u, we can construct two vertex-disjoint paths P_1 and P_2, which cover all vertices of X_n, while P_1 joins s and u, and P_2 joins t and v.

Lemma 4. *If X_{n-1} is (n-4) edge Hamiltonian laceable and (n-3) edge Hamiltonian, then X_n is (n-3) edge Hamiltonian laceable, for n > 4.*

Proof. Let s and t be two arbitrary vertices such that s is red and t is blue. And let F be the faulty edge set of X_n with $|F| = n$-3.

Case 1. *$F \subset X_{n-1}^0$ or $F \subset X_{n-1}^1$*

Without loss of generality, we can assume that $F \subset X_{n-1}^1$.

Case 1.1. $s, t \in X_{n-1}^0$

Since X_{n-1}^1 is (n-3) edge Hamiltonian, there exists Hamiltonian cycle C_1 in $X_{n-1}^1 - F$. Let $(\phi(u), \phi(v))$ be an edge of C_1, where $\phi(u)$ is blue and $\phi(v)$ is red. We can write the cycle C_1 as $\langle \phi(u) \rightarrow P_0 \rightarrow \phi(v), \phi(u)\rangle$. Applying **Lemma 3**, we can construct two spanning vertex-disjoint paths $P(s, u)$ and $P(v, t)$ in X_{n-1}^0, where u is a blue node and v is a red node. Therefore, there exists a Hamiltonian path $\langle s \rightarrow P(s, u) \rightarrow u, \phi(u) \rightarrow P_1 \rightarrow \phi(v), v \rightarrow P(v, t) \rightarrow t\rangle$ in X_n, as illustrated in Fig. 3(a).

Case 1.2. $s \in X_{n-1}^0$ *and* $t \in X_{n-1}^1$

Since X_{n-1}^1 is (n-3) edge Hamiltonian, there exists a Hamiltonian cycle C_1, denote as $\langle t \rightarrow P_1 \rightarrow \phi(p), t\rangle$, in X_{n-1}^1, where $(t, \phi(p))$ is an edge of C_1. In X_{n-1}^0, there exists a Hamiltonian path P_0 from s to p. Thus, $\langle s \rightarrow P_0 \rightarrow p, \phi(p) \rightarrow P_1 \rightarrow t\rangle$ forms a Hamiltonian path from s to t, as illustrated in Fig. 3(b).

Case 1.3. $s, t \in X_{n-1}^1$

Since X_{n-1}^1 is (n-4) edge Hamiltonian laceable, there exist two spanning vertex-disjoint paths $P(s, u)$ and $P(v, t)$ for some edge (u, v) in $X_{n-1}^1 - F$. In X_{n-1}^0, there exists a Hamiltonian path P_0 from $\phi(u)$ to $\phi(v)$. Thus, $\langle s \rightarrow P(s, u) \rightarrow u, \phi(u) \rightarrow P_0 \rightarrow \phi(v), v \rightarrow P(v, t) \rightarrow t\rangle$ forms a Hamiltonian path from s to t, as illustrated in Fig. 3(c).

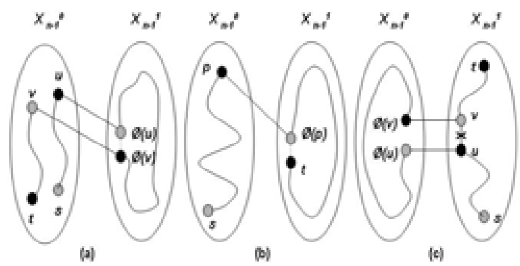

Fig. 3. (a) Case 1.1: $s, t \in X_{n-1}^0$ (b) Case 1.2: $s \in X_{n-1}^0$, $t \in X_{n-1}^1$ (c) Case 1.3: $s, t \in X_{n-1}^1$

Case 2. *$F \not\subset X_{n-1}^0$ and $F \not\subset X_{n-1}^1$*

Since $|F \cap X_{n-1}^0| \leq n - 4$ and $|F \cap X_{n-1}^1| \leq n -$

4, both $X_{n-1}^0 - F$ and $X_{n-1}^1 - F$ are Hamiltonian laceable.

Case 2.1. *s and t are in the same subgraph.*

Without loss of generality, we can assume that $s, t \in X_{n-1}^0$. Since X_{n-1}^0 is $(n$-4) edge Hamiltonian laceable, there exists a Hamiltonian path P_0 from s to t with $2^{n-1} - 1$ edges. Since $\lceil (2^{n-1} - 1)/2 \rceil > n - 3$ for $n > 4$, there exists an edge (u, v) in path P_0, such that both edges $(u, \phi(u))$ and $(v, \phi(v))$ are fault-free. Since X_{n-1}^1 is $(n$-4) edge Hamiltonian laceable, there exists a Hamiltonian path P_1 from $\phi(u)$ to $\phi(v)$. We can construct the Hamiltonian from s to t as $\langle s \rightarrow P(s, u) \rightarrow u, \phi(u) \rightarrow P_1 \rightarrow \phi(v), v \rightarrow t \rangle$, as illustrated in Fig. 4(a).

Case 2.2. *s and t are in different subgraphs.*

Without loss of generality, we can assume that $s \in X_{n-1}^0$ and $t \in X_{n-1}^1$. Let z be a blue vertex in X_{n-1}^0 such that $(z, \phi(z))$ is fault-free. Since $X_{n-1}^0$and X_{n-1}^1 is $(n$-4) edge Hamiltonian laceable. There exist Hamiltonian paths $P(s, z)$ and $P(\phi(z), t)$ in $X_{n-1}^0$and X_{n-1}^1, respectively. We can construct the Hamiltonian path from s to t as $\langle s \rightarrow P(s, z) \rightarrow z, \phi(z) \rightarrow P(\phi(z), t) \rightarrow t \rangle$, as illustrated in Fig. 4(b). □

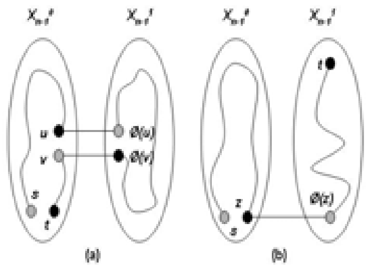

Fig. 4. (a) Case 2.1: $s, t \in X_{n-1}^0$ (b) Case 2.2: $s \in X_{n-1}^0$ and $t \in X_{n-1}^1$

Lemma 5. *If X_{n-1} is (n-4) edge Hamiltonian laceable and (n-3) edge Hamiltonian, then X_n is (n-2) edge Hamiltonian, for n > 3.*

Proof. Let F be the faulty edge set in X_n with $|F| = n$ - 2. Both two subgraphs X_{n-1}^0 and X_{n-1}^1 are $(n$-4) edge Hamiltonian laceable and $(n$-3) edge Hamiltonian.

Case 1. $F \subset X_{n-1}^0$ or $F \subset X_{n-1}^1$

Without loss of generality, we can assume that $F \subset X_{n-1}^1$. Since X_{n-1}^1 is $(n$-3) edge Hamiltonian, there exists a Hamiltonian path $P(x, y)$ in $X_{n-1}^1 - F$ for some edge (x, y) of X_{n-1}^1. There also exists a Hamiltonian path $P(\phi(x), \phi(y))$ between $\phi(x)$ and $\phi(y)$ in X_{n-1}^0. Thus, $\langle x \rightarrow P(x, y) \rightarrow y, \phi(y) \rightarrow P(\phi(y), \phi(x)) \rightarrow \phi(x)\ , x \rangle$ forms a Hamiltonian cycle of $X_n - F$, as illustrated in Fig. 5(a).

Case 2. $F \not\subset X_{n-1}^0$ and $F \not\subset X_{n-1}^1$

Without loss of generality, we can assume that $|F_0| \leq |F_1|$. Thus, $X_{n-1}^0 - F_0$ is Hamiltonian laceable and $X_{n-1}^1 - F_1$ is Hamiltonian. Let C_1 be the Hamiltonian cycle in $X_{n-1}^1 - F$. Since $\lceil (2^{n-1}-1)/2 \rceil > n - 2$ for $n > 3$, there exists an edge (x, y) of C_1 such that both $(x, \phi(x))$ and $(y, \phi(y))$ are fault-free. We can write the cycle C_1 as $\langle x \rightarrow P(x, y) \rightarrow y, x \rangle$. Since $X_{n-1}^0 - F$ is Hamiltonian laceable, there exists a Hamiltonian path $P(\phi(y), \phi(x))$ in $X_{n-1}^0 - F_0$. Hence, $\langle x \rightarrow P(x, y) \rightarrow y, \phi(y) \rightarrow P(\phi(y), \phi(x)) \rightarrow \phi(x), x \rangle$ forms a Hamiltonian cycle in $X_n - F$, as illustrated in Fig. 5(b). □

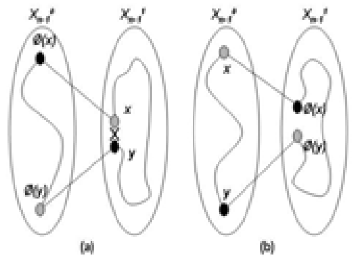

Fig. 5. (a) Case 1: $F \subset X_{n-1}^1$ (b) Case 2: $F \subset X_{n-1}^0$ and X_{n-1}^1

Theorem 2. The n-dimensional bipartite HL-graph X_n *is (n-3) edge Hamiltonian laceable, and (n-2) edge Hamiltonian, for $n \geq 3$.*

Proof. We will prove this theorem with induction on n. The base case: **Lemma 1** shows X_3 is 1 edge Hamiltonian laceable. Thus X_3 is also 1 edge Hamiltonian. We show X_4 is 1 edge Hamiltonian in **Lemma 2**. Applying **Lemma 5**, we can prove X_4 is 2 edges Hamiltonian. The induction step can be proved with **Lemma 4** and **Lemma 5**. □

4. Concluding remarks

In [5], Tsai et al. showed that the hypercube Q_n is $(n$-2) edge Hamiltonian laceable and $(n$-3) edge hyper-Hamiltonian laceable. The HL-graph is more general than hypercube graph. In this paper, we show that every *n*-dimensional bipartite HL-graph is hyper-Hamiltonian laceable, (*n*-3) edge Hamiltonian laceable and (*n*-2) edge Hamiltonian. It is worthwhile to investigate other fault tolerance Hamiltonian properties of hypercube-like graphs.

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