通路圖形及通路數值
Rearrangeable Graphs and Rearrangeability Numbers

吳致中 王有禮 王福星

Chih–Chung Wu Yue–Li Wang* Fu–Hsing Wang

Department of Information Management, National Taiwan University
of Science and Technology, Taipei, Taiwan, Republic of China.

Abstract

Given a directed graph $D(V, A)$, where $V = \{1, 2, 3, \ldots, n\}$ is the vertex set and A is the directed arc set. A permutation $\sigma = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ \pi_{v_1} & \pi_{v_2} & \ldots & \pi_{v_n} \end{pmatrix}$ is said to be realizable on D if there exist n arc-disjoint paths in D that connect vertex v_i to vertex π_{v_i},$i = 1, 2 \ldots, n$. D is said to be rearrangeable if all $n!$ permutations are realizable on D. Moreover, the rearrangeability number of D is the minimum multiplicity that every arc needs to be duplicated in order for D to become rearrangeable. In this paper, we prove that complete n-partite digraphs are rearrangeable graphs. Then, we give the upper and lower bounds of the rearrangeability number of torus networks.

摘 要

在有向圖G中， 對於G之一排列π， 若在G中存在n條從i到$\pi(i)$,$1 \leq i \leq n$, 兩兩皆爲邊獨立之路徑, 則稱π爲realizable。若所有$n!$個排列皆爲 realizable, 則稱G爲一個通路圖形。另一方面, 要使G爲一通路圖形, 其每條邊所需複製的最少次數, 稱爲G的通路數值。在本篇論文中, 將證明完全多分有向圖爲通路圖形。並證明圓盤網路的通路數值之上、下限。

Keyword: Rearrangeable graphs, Rearrangeability number, Complete n-partite digraphs, Torus network.

*All correspondence should be addressed to Professor Yue–Li Wang, Department of Information Management, National Taiwan University of Science and Technology, 43, Section 4, Kee-Lung Road, Taipei, Taiwan, Republic of China (Phone: 886–02–27376768, Fax: 886–02–27376777, Email: ylwang@cs.ntust.edu.tw).

1 Introduction

The topology of a multiprocessor system can be, in general, modeled by a directed graph, where each vertex represents a processor and each arc represents the communication link between two processors. Let $D = (V, A)$ be a directed graph(digraph), where $V = V(D) = \{1, 2, \ldots, n\}$ and $A \subseteq V \times V$. Let $\sigma = \left(\begin{array}{cccc} v_1 & v_2 & \ldots & v_n \\ \pi_{v_1} & \pi_{v_2} & \ldots & \pi_{v_n} \end{array} \right)$ be an *n-permutation* of n symbols $v_1, v_2, \ldots, v_n$. An n-permutation σ is called *realizable* on D if n arc disjoint paths can be established in D that connect v_i to π_{v_i}, for $i = 1, 2 \ldots, n$, respectively. Digraph D is *rearrangeable* if all $n!$ n-permutations are realizable in D. Research on the rearrangeability are widely studied on networks, such as hypercubes, Benes Networks and Omega Networks [1, 5, 8, 7, 9]. In [3], Hu et al. showed that complete digraphs and stars are rearrangeable. A related problem is to make a digraph rearrangeable by duplicating arcs. The times of the duplicity is said to be the *multiplicity.* Let D^m be the *m-multiple digraph* of D which is obtained by duplicating every arc with multiplicity m; notice that D^1=D. The *rearrangeability number* $\psi(D)$ is the minimum value of m such that D^m is rearrangeable. Let $\psi(D)$=$+\infty$ if D is not strongly connected. Hu et al.[3] gave the bounds of rearrangeability number in trees, rings, meshes and hypercubes.

Let $\sigma = \left(\begin{array}{cccc} v_1 & v_2 & \ldots & v_n \\ \pi_{v_1} & \pi_{v_2} & \ldots & \pi_{v_n} \end{array} \right)$ be a permutation with n entries, $\left(\begin{array}{c} v_i \\ \pi_{v_i} \end{array} \right)$, for $i = 1, 2, \ldots, n$. A permutation σ' is a *subpermutation* of a permutation σ if σ' is obtained by removing some entries from σ. A permutation is called *dearrangement* if $\pi_{v_i} \neq v_i$, for $i = 1, 2, \ldots, n$. A permutation σ' is a *dearrangement subpermutation* of a permutation σ if it is obtained from σ by removing the entries $\pi_{v_i} = v_i$. There are two rows of symbols in a permutation. Such a symbol we call it as *incident vertex.* Let set *incident*(σ') be the set of incident vertices of the subpermutation σ'. For example, the permutation $\sigma = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 4 & 3 & 1 \end{array} \right)$ has π_1=5, π_2=6, π_4=4, etc. Permutations $\left(\begin{array}{cccc} 1 & 2 & 4 & 6 \\ 5 & 6 & 4 & 1 \end{array} \right)$, $\left(\begin{array}{ccccc} 1 & 2 & 3 & 5 & 6 \\ 5 & 6 & 2 & 3 & 1 \end{array} \right)$ are subpermutations of σ. Then, $\sigma' = \left(\begin{array}{ccccc} 1 & 2 & 3 & 5 & 6 \\ 5 & 6 & 2 & 3 & 1 \end{array} \right)$ is the dearrangement subpermutation of σ, where incident$(\sigma') = \{1, 2, 3, 5, 6\}$. Clearly, a permutation is realizable if its dearrangement subpermutation is realizable. For simplicity, we assume that the given permutations in this paper are all dearrangement.

In this paper, we shall show that complete n-partite digraphs are rearrangeable. For the purpose of practical applications on interconnection networks, torus network is studied which is an alternative of mesh. The upper and lower bounds of rearrangeability number of torus networks is given.

The remaining part of this paper is organized as follows. In the next section, the complete bipartite digraphs and complete n-partite digraphs are shown to be rearrangeable. In Section 3, the upper and lower bounds of rearrangeability number of torus networks are established. Finally, this paper concludes with some remarks in Section 4.

2 Complete n-partite digraphs are rearrangeable

A *complete n-partite digraph* $K_{m_1,m_2,\ldots,m_n}$ is a digraph whose vertex set can be partitioned into n partite sets such that the arcs $< u, v >$ and $< v, u >$ exist if and only if u and v are the vertices of two different partite sets. Indeed, if n=2, then we name it complete bipartite digraph which is denoted by $K_{m,n}$. Since stars are rearrangeable[3], the complete bipartite digraph $K_{m,n}$ is rearrangeable when m=1 or n=1.

Let $S \subseteq V(K_{m,n})$, we define $\Pi(S) = \bigcup_{v\in S}\{\pi_v\}$. A *path* P is a sequence $v_1e_1v_2e_2v_3 \ldots v_ke_kv_{k+1}$, denoted by $v_1 \to v_2 \to v_3 \to \ldots \to v_k \to v_{k+1}$, with elements alternately from the vertex set V and the arc set A of a digraph D where $v_i \neq v_j$ for $i \neq j$. Vertex v_1 is called the *source* vertex of P, and vertex v_{k+1} is the *destination* vertex of P. Two paths are called *disjoint* if there are no common arc in them.

Lemma 1 *Complete bipartite digraphs $K_{m,n}$ are rearrangeable, for $m, n \geq 2$.*

Proof. Let $V_1 = \{a_1, a_2, \ldots, a_m\}$ and $V_2 = \{b_1, b_2, \ldots, b_n\}$ be the partite sets of $K_{m,n}$. Since $|V(K_{m,n})| = m + n$, we shall give an algorithm to construct the $m + n$ disjoint paths that connect v_i to π_{v_i}, $v_i \in V_1 \bigcup V_2$, for each one of the $(m + n)!$ permutations.

If there exists $\pi(a_i) = b_j$ or $\pi(b_j) = a_i$, then the path connects the pair of vertices a_i and b_j is simply the arc $< a_i, b_j >$ if $\pi(a_i) = b_j$ or $< b_j, a_i >$ if $\pi(b_j) = a_i$. Notice that both arcs $< a_i, b_j >$ and $< b_j, a_i >$ exist in a complete bipartite digraph. For $\pi(a_i) = a_j$(respectively, $\pi(b_i) = b_j$), $i \neq m$, the path $a_i \to b_1 \to a_j$(respectively, $b_i \to a_1 \to b_j$) is chosen. Finally, for $\pi(a_m) = a_i$(respectively, $\pi(b_m) = b_i$), if it exists, the path $a_m \to b_2 \to a_i$(respectively, $b_m \to a_2 \to b_i$) is chosen. Clearly, the above $m + n$ paths are disjoint and the lemma follows.

Q. E. D.

For example, $V_1 = \{1,2,3,4\}$ and $V_2 = \{5,6,7,8,9\}$ are the two partite sets of $K_{4,5}$. The 9 disjoint paths for permutaion $\left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 4 & 2 & 3 & 7 & 1 & 9 & 8 \end{array}\right)$ are listed in the following.

$1 \to 5$

$2 \to 6$

$3 \to 5 \to 4$

$4 \to 6 \to 2$

$5 \to 3$

$6 \to 1 \to 7$

$7 \to 1$

$8 \to 1 \to 9$

$9 \to 2 \to 8$

Since complete bipartite digraphs $K_{m,n}$ are rearrrangeable, there are $m+n$ disjoint paths that connect v_i to π_{v_i}, $i = 1,2\ldots,m+n$, for each permutation σ. It is clear, for each subpermutation $\sigma' = \left(\begin{array}{cccc} u_1 & u_2 & \ldots & u_k \\ \pi_{u_1} & \pi_{u_2} & \ldots & \pi_{u_k} \end{array}\right)$ of σ, there are k disjoint paths that connect u_i to π_{u_i}, $i = 1,2,\ldots,k$. Furthermore, σ' is realizable.

Corollary 2 *Every subpermutation with k entries contains k disjoint paths on a complete bipartite digraph.*

For each two distinct partites V_i, V_j of a complete n-partite digraph, the induced subgraph of $V_i \bigcup V_j$ is a complete bipartite digraph which is called *induced complete bipartite digraph $B_{i,j}$* of $K_{m_1,m_2,\ldots,m_n}$. Let $\Sigma = m_1 + m_2 + \ldots + m_n$.

Theorem 3 *Complete n-partite digraphs $K_{m_1,m_2,\ldots,m_n}$ are rearrangeable.*

Proof. Let $V_i = \{v_{i,k} | 1 \le k \le m_i\}$, for $i = 1,2,\ldots,n$. The vertex set $V(K_{m_1,m_2,\ldots,m_n}) = \bigcup_{i=1,2,\ldots,n} V_i$ where $V_j \bigcap V_k = \emptyset$, for $j \neq k$. Let $V_i = W_{i,1} \bigcup W_{i,2} \bigcup \ldots \bigcup W_{i,n}$, $i = 1,2,\ldots,n$, where $W_{i,k} = \{v \in V_i | \pi_v \in V_k\}$, for $k = 1,2,\ldots,n$. If $\Pi(V_i) \bigcap V_k = \emptyset$, then $W_{i,k} = \emptyset$. For any permutation $\sigma = \left(\begin{array}{cccc} 1 & 2 & \ldots & \Sigma \\ \pi_1 & \pi_2 & \ldots & \pi_\Sigma \end{array}\right)$, we split σ into subpermutations so that for each subpermutation σ', incident(σ')$\subseteq V(B_{i,j})$. By Lemma 1, we construct the disjoint paths of each $B_{i,j}$. By Corollary 2, the paths for each subpermutation can be found in its corresponding induced complete bipartite digraph. Thus, the complete n-partite digraphs $K_{m_1,m_2,\ldots,m_n}$ are rearrangeable.

Q. E. D.

Take complete 3-partite digraph $K_{3,3,3}$ as an example. Let $V(K_{3,3,3}) = V_1 \bigcup V_2 \bigcup V_3$, where $V_1 = \{1,2,3\}, V_2 = \{4,5,6\}, V_3 = \{7,8,9\}$. For a permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 4 & 2 & 7 & 6 & 1 & 8 & 3 & 5 \end{pmatrix}$, we split it into three subpermutations $\alpha = \begin{pmatrix} 2 & 3 & 6 \\ 4 & 2 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 7 & 8 \\ 9 & 8 & 3 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 4 & 5 & 9 \\ 7 & 6 & 5 \end{pmatrix}$. By Lemma 1 and Corollary 2, we construct 3 disjoint paths in $B_{1,2}$ for the subpermutation α. There are also three disjoint paths been established in $B_{1,3}$ and $B_{2,3}$ for the subpermutation α, β, respectively.

3 Torus networks

Let M be an $n \times n$ mesh with n^2 vertices and (r, c) be the vertex in row r and column c. A *torus* T is a mesh with wrap-around arcs in the rows and columns. Figure 1 shows a mesh and a torus. Notice that two opposite directed arcs are assumed between two adjacent vertices. In an $n \times n$ mesh, there are $2n$ directed arcs between two adjacent rows(columns). The rows from 1 to $\lfloor \frac{2}{n} \rfloor$ are the upper half of an $n \times n$ mesh and the rest is lower half of the mesh. If n is even, Hu et al.[3] constructed a permutation π such that every vertex in the upper(lower) half of the mesh is mapped to a unique vertex in the lower(upper) half of the mesh. To be rearrangeable, the $2n$ directed arcs between the two parts should be duplicated at least $\frac{n^2}{2n} = \frac{n}{2}$ times. Hence, $\psi(M) \geq \frac{n}{2}$. If n is odd, the half with smaller size consists of $\lfloor \frac{n}{2} \rfloor \times n$ vertices. Hu et al.[3] made a permutation such that $\lfloor \frac{n}{2} \rfloor \times n$ vertices in each half are mapped to the other half. So, $\psi(M) \geq \frac{\lfloor \frac{n}{2} \rfloor \times n \times 2}{2n} = \lfloor \frac{n}{2} \rfloor$. Thus, $\psi(M) \geq \lfloor \frac{n}{2} \rfloor$.

To show the upper bound of the rearrangeability number of a mesh, a realization is made. Let $k = (r, c)$ and $\pi_k = (r', c')$ where $r' \geq r$ and $c' \geq c$. For source vertex (r, c), let the path pass through (r, c') to the destination vertex (r', c'). In this realization, the rightmost arc $< (r, n-1), (r, n) >$ in row r will be used at most $n - 1$ times. The same holds for the arcs $< (r, 1), (r, 2) >, < (1, c), (2, c) >$ and $< (n-1, c), (n, c) >$. Thus, Hu et al.[3] gave the upper bound $n - 1$.

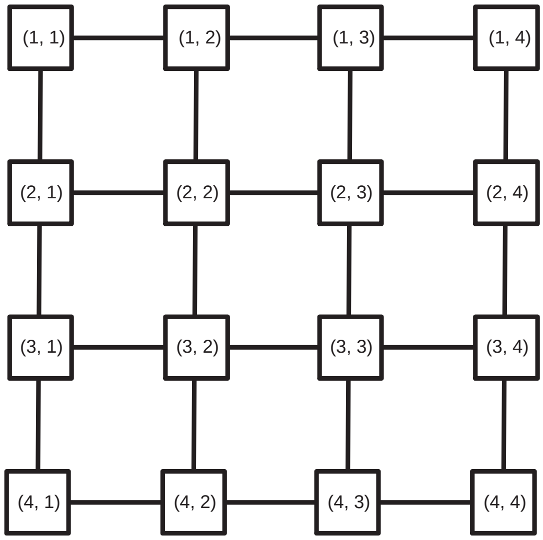
Theorem 4 $\lfloor \frac{n}{2} \rfloor \leq \psi(M) \leq n - 1$ *for an* $n \times n$ *mesh* M*[3].*

In an $n \times n$ torus network T, the wrap-around arcs exist in the rows and columns. There are $4n$ directed arcs between the two halves. Based on the algorithms of Hu et al., $\psi(T) \geq \frac{n^2}{4n} = \frac{n}{4}$, if n is even. And $\psi(T) \geq \frac{\lfloor \frac{n}{2} \rfloor \times n \times 2}{4n} = \lfloor \frac{n}{4} \rfloor$, if n is odd. Thus, $\psi(T) \geq \lfloor \frac{n}{4} \rfloor$. Then, we want to show the upper bound of $\psi(T)$. Let $k = (r, c)$ and $\pi_k = (r', c')$. If $c = 1$ and $c' = n$, then the path oriented from $(r, 1)$ pass through the wrap-around arc $< (r,1),(r,n) >$ to the destination vertex (r',n). Otherwise, For source vertex (r,c), $2 \leq c \leq n-1$, let the path pass through (r,c') to the destination vertex (r',c'). In this realization, the rightmost arc $< (r,n-1),(r,n) >$ in row r will be used at most $n-2$ times. The same holds for the arcs $< (r,1),(r,2) >, < (1,c),(2,c) >$ and $< (n-1,c),(n,c) >$. Thus, $\psi(T) \leq n-2$. Therefore, we immediately have the next theorem.

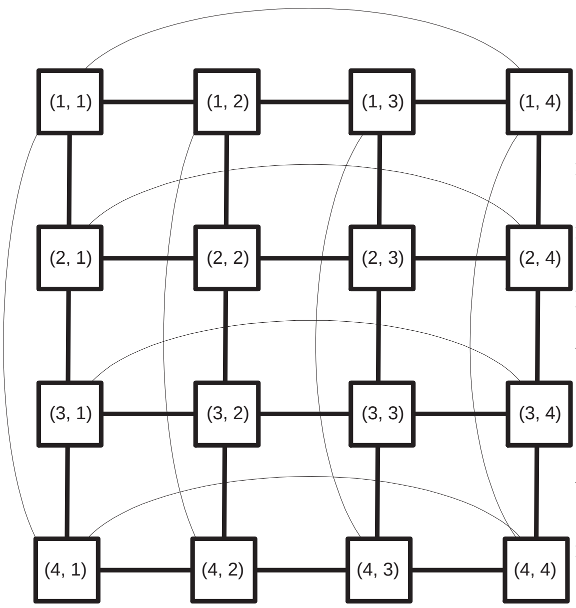
Theorem 5 $\lceil \frac{n}{4} \rceil \leq \psi(T) \leq n-2$ *for an* $n \times n$ *torus* T.

4 Concluding Remarks

Directed interconnection networks have gained much attention in the recent research in interconnection networks. In this paper, we show that complete n-partite directed graphs are rearrangeable. We also give the lower and upper bounds of the rearrangeability number of torus networks. In the future research, the rearrangeability of the interconnection networks, for example, Butterfly networks, MultiMesh networks, etc., are interesting to study.



(a)



(b)

Figure 1: (a)A 4×4 mesh;(b)a 4×4 torus.

References

[1] V.E. Benes, The Mathematical Theory of Connecting Networks and Telephone Traffic, Aca-

demic Press, New York, 1965.

[2] M.R. Garey, D.S. Johnson, Computers and Intractability : A Guide to the Theory of NP-Completeness, W.H. Freeman, San Francisco, CA, 1979.

[3] Q. Hu, Y. Ahang, X. Shen, Rearrangeable graphs, Information Processing Letters, 71 (1999), 23-27.

[4] Q. Hu, Y. Zhang, X. Shen, Rearrangeable graphs, in: 3rd Annual International Conference, COCOON'97, Shanghai, China, August 20-22, Lecture Notes in Computer Science 1276, Springer, Berlin, 1997, 441-449.

[5] K. Hwang, F.A. Briggs, Computer Architecture and Parallel Processing, McGraw-Hill, New York, 1984.

[6] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes, Morgan Kaufmann, San Mateo, CA, 1992.

[7] X. Shen, Q. Hu, Realizability of an arbitrary permutation on a circuit-switched hypercube, Information Processing Letters 51 (1994), 237-243.

[8] H.J. Siegel, Interconnection Networks for Large-Scale Parallel Processing, McGraw-Hill, New York, 1990.

[9] T. Szymanski, On the permutation capability of a circuit-switched hypercube, in: Proceedings of International Conference on Parallel Processing, (1989), I:103-110.