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GLOBAL CONVERGENCE OF HOPFIELD NETWORKS IN PARTIAL SIMULTANEOUS UPDATE MODE

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ABSTRACT

This paper presents new stability conditions for Hopfield networks in partial simultaneous updating (PSU) mode. First, an example is given to demonstrate that oscillation may occur even if one use a connection matrix satisfying the conditions derived in [3]. Then, new sufficient conditions ensuring global convergence of a Hopfield network in PSU mode are given. The obtained results permit a little relaxation on the lower bound of the main diagonal elements of the connection matrix.

1.INTRODUCTION

The Hopfield network [1],[2] is one of the famous neural networks with a wide range of applications. In the synthesis of such a network, ensuring a convergence of the state trajectories starting from arbitrary initial state to a fixed point is of particular importance. Such a convergence property is the basis for the potential applications of the network, such as content addressable memory [2], pattern recognition [1], and combinatorial optimization [10]. Afterwards many researchers have focused on the following two distinct update modes: 1) asynchronous (or serial) mode, in which a neuron is chosen at random and then its value is updated, and 2) synchronous (or fully parallel) mode, where all of the neurons are simultaneously updated. Sufficient conditions for global convergence of above two update modes have been extensively studied [1],[4],[10]-[12]. However, the characteristic of updating a partial group of neurons has received little attention in previous literatures. Cernuschi-Frias [3] has presented a "macroneuron" concept. He considers updating simultaneously groups of a fixed number of neurons. Each of these groups is referred to as a "macroneuron" in [3]. The sufficient conditions on the corresponding connection matrix as to ensure global stability have been derived. However, it is found by our experiment that, even with a connection matrix satisfying the conditions in [3], the state of a Hopfield network may converge to limited cycles. It means that oscillation may occur.

In this paper, the concept of partial simultaneous updating (PSU) in Hopfield network is first reviewed. An example is given to demonstrate the oscillation phenomenon described above. Then, by means of the results derived in [5], we give a new condition which ensures global convergence of a Hopfield network in PSU mode. The difference of the obtained results with those derived in [4] is demonstrated by another example.

2. REVIEW OF PREVIOUS RESULTS

Consider a Hopfield network consisting of n fully connected two-state neurons $X \in \{-1, 1\}^n$. Let $T = [t_{ij}]$ denote the connection matrix of this network. Then the asynchronous updating cycle of the network is determined by the following equation:

$$x_i' = sgn\left\{\sum_{j=1}^{n} t_{ij}x_j - \theta_i\right\}, \qquad (1)$$

in which x_i is the ith component of X; x_i' denotes the next state of x_i; t_{ij} is the connection weight from neuron j to neuron i; θ_i represents the threshold attached to neuron i (for simplicity, hereinafter we let $\theta_i = 0 \forall i$); $sgn\{\alpha\} = 1$ for $\alpha \geq 0$; $sgn\{\alpha\} = -1$ otherwise. The updating cycle is asynchronous in the sense that neuron states are updated one at a time by following (1) with equal probabilities. The macroneuron concept proposed in [3] is obtained by considering x_i as a column vector X_i (or a macroneuron) which has a fixed number q_i of components which takes values +1 or −1. Each element t_{ij} is thus considered to be a $q_i \times q_j$ matrix T_{ij}, with $(k,h)th$ element $T_{ij}(k,h)$.

Let M be the number of macroneurons. Then the total number of neurons is given by

$$N = \sum_{j=1}^{M} q_j. \tag{2}$$

The ith macroneuron is updated according to

$$X_i' = sgn\left\{\sum_{j=1}^{M} T_{ij}X_j\right\}; \tag{3}$$

specifically, each neuron is updated as

$$X_i'(k) = sgn\left\{\sum_{j=1}^{M}\sum_{h=1}^{q_j} T_{ij}(k,h)X_j(h)\right\}. \tag{4}$$

Since a group of neurons is updated each time, the Hopfield network is said to be operated in the partial simultaneous updating (PSU) mode. Define the matrix T to be composed of the blocks $T_{ij}s$. Then, the author in [3] proves that a Hopfield network in PSU mode is globally stable if the matrix T satisfies

i) $T_{ij} = (T_{ji})^T$ for all $i \neq j$,

ii) T_{ii} : nonnegative definite but not necessarily symmetric.

In other words, the matrix T is not necessarily symmetric. As shown in [3] the macroneuron concept permits a little relaxation on the symmetry hypothesis of the connection matrix [1]. However, it is found by our experiment that, even with a matrix T that satisfies conditions i) and ii), the state of a Hopfield network may converge to limited cycles (see next section). That is, undesired oscillation may occur.

3. STABILITY OF HOPFIELD NETWORK IN PSU MODE

With a connection matrix satisfying conditions i) and ii), the state of a Hopfield network may converge to limited cycles. It means that a PSU sequence may not converge to fixed points. Following is a counter-example to conditions i) and ii).

Example 1:

Consider a Hopfield network with six neurons ($N = 6$) and the following connection matrix,

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{5}$$

in which

$$T_{11} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 4 & -2 \\ -4 & 2 & 1 \end{bmatrix},$$

$$T_{12} = (T_{21})^T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix},$$

$$T_{22} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & -4 \\ -3 & 4 & 2 \end{bmatrix}.$$

Note that in this case there are only two macroneurons ($M = 2$, $q_1 = q_2 = 3$) and the system is structural the same as the Intraconnected Bidirectional Associative Memory (IBAM, [8],[9]). The main difference is that the latter uses the Outer Product Rule [8] to encode matrices T_{11}, T_{12}, and T_{22}. Now (3) can be rewritten as

$$\left\{ \begin{array}{l} X_1' = sgn\{T_{11}X_1 + T_{12}X_2\} \\ X_2' = sgn\{T_{21}X_1' + T_{22}X_2\}. \end{array} \right\} \tag{6}$$

It can be easily checked that T_{11} and T_{22} are both positive definite. However, let $X_1^0 = (-1,-1,1)^T$ and $X_2^0 = (1,-1,-1)^T$ be initial states of the network, the following sequence can be obtained,

$$X_1' = (-1,1,1)^T; X_2' = (1,-1,1)^T;$$

$$X_1'' = (-1,1,-1)^T; X_2'' = (-1,1,1)^T;$$

$$X_1^{(3)} = (1,1,-1)^T; X_2^{(3)} = (-1,1,-1)^T;$$

$$X_1^{(4)} = (1, -1, -1)^T; X_2^{(4)} = (1, 1, -1)^T;$$

$$X_1^{(5)} = (1, -1, 1)^T; X_2^{(5)} = (1, 1, -1)^T;$$

$$X_1^{(6)} = (-1, -1, 1)^T = X_1^0;$$

$$X_2^{(6)} = (1, -1, -1)^T = X_2^0;$$

$$\vdots$$

From above observation, we can conclude that $X_1^{(t)} = X_1^{(t-6)}$, $X_2^{(t)} = X_2^{(t-6)}$, for $t \geq 6$
It is obvious that a limited cycle of period (length) 6 occurs. Moreover, we test the network with all 64 ($2^6 = 64$) possible bipolar states in the state space. It is found that all 64 states converge to the limited cycle described above. We can thus summarize that the state of a Hopfield network may converge to a limited cycle even if condition i) and ii) hold. In fact, from this example there is no fixed point within the entire state space of the network.

4. NEW STABILITY CONDITIONS

In [5],[6] sufficient conditions ensuring global convergence of a Hopfield network in serial and fully parallel modes are both given. New stability conditions for a Hopfield network in PSU mode can be derived on the basis of these results. First we examine the following lemma.
Lemma 1 (Xu & Kwong [5]):
Let $T = [t_{ij}]$ be the connection matrix of a Hopfield network, T not necessarily symmetric. Assume the network is operating in fully parallel mode. Then,

$$E(X) = -\frac{1}{2} X^T T X \qquad (7)$$

will be a strict Liapunov function of the network (*i.e.*, $E(X^{(t+1)}) < E(X^{(t)}$ for any $X^{(t+1)} \neq X^{(t)}$). Moreover, let $\Delta E(t) = E(X^{(t+1)}) - E(X^{(t)})$, $\Delta X(t) = X^{(t+1)} - X^{(t)}$, and $I(t) = \{i \in \{1, 2, ..., N\} : \Delta x_i(t) \neq 0\}$, then

$$\Delta E(t) \leq -\frac{1}{2} \sum_{i \in I(t)} \sum_{j \in I(t)} t_{ij}^* \Delta x_i(t) \Delta x_j(t), \qquad (8)$$

where

$$t_{ij}^* = \left\{ \begin{array}{ll} t_{ii} - \frac{1}{2} \sum_{k=1}^{N} |t_{ki} - t_{ik}|, & \text{if } i = j \\ t_{ij} & \text{if } i \neq j \end{array} \right\}.$$

Proof:
Please refer to Theorem 2 of [5] (let the threshold $t_i = 0 \forall i$).
Using Lemma 1 one can derive the following theorem.
Theorem 1:
Define the matrix T to be composed of the blocks $T_{ij}s$. If

$$T_{ii}^* (i = 1, ..., M) \text{ is nonnegative definite}, \qquad (9)$$

where, for each element in the block T_{ii}^*,
$T_{ij}^*(k, h) =$

$$\left\{ \begin{array}{l} T_{ij}(k,h), \text{ if } i \neq j \\ T_{ii}(k,k) - \frac{1}{2} \sum_{u=1}^{M} \sum_{l=1}^{q_u} |T_{ui}(l.k) - T_{iu}(k,l)|, \\ \text{ if } i = j \ \& \ k = h, \\ T_{ii}(k,h), \text{ if } i = j \ \& \ k \neq h \end{array} \right\} \qquad (10)$$

then the Hopfield network with connection matrix T globally converges to a stable state when operating in the PSU mode.
Proof:
From Lemma 1, and assume only macroneuron X_p is selected to be updated, we have
$\Delta E(t)$

$$\begin{array}{rl} \leq & -\frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k \in I(t)} \sum_{h \in I(t)} T_{ij}^*(k,h) \Delta X_i(k) \Delta X_j(h) \\ = & -\frac{1}{2} \sum_{k \in I(t)} \sum_{h \in I(t)} T_{pp}^*(k,h) \Delta X_p(k) \Delta X_p(h) \end{array} \qquad (11)$$

Global convergence of a Hopfield network can be ensured by showing that $\Delta E(t)$ whenever $\Delta X(t) \neq 0$. From (11) it is obvious that $\Delta E(t) \leq 0$ if T_{pp}^* is nonnegative definite. Since any one of the M macroneurons is possible to be selected within the entire PSU sequence, global convergence is guaranteed if

$$T_{ii}^* \text{ is nonnegative definite for all } i (i = 1, ..., M).$$

This completes the proof.
A sufficient condition ensuring (9) is given by the following corollary.

Corollary 1:
A Hopfield network operating in PSU mode is globally stable if the elements of the corresponding matrix T satisfy

$$\begin{aligned} T_{ii}(k,k) &\geq \frac{1}{2}\left\{\sum_{u=1}^{M}\sum_{l=1}^{q_u}|T_{ui}(l,k)-T_{iu}(k,l)|\right\} \\ &\quad + \left\{\sum_{h\neq k,h=1}^{q_i}|T_{ii}(k,h)+T_{ii}(h,k)|\right\} \\ i &= 1,...,M; \\ & k=1,...,q_i. \end{aligned} \quad (12)$$

Proof:
Note that T^*_{ii} is not necessary symmetric. Condition (9) can be reformulated by replacing T^*_{ii} with its symmetric component $S = \frac{1}{2}\{T^*_{ii} + (T^*_{ii})^T\}$. This can be demonstrated by decomposing T^*_{ii} into S and its antisymmetric component $A = \frac{1}{2}\{T^*_{ii} - (T^*_{ii})^T\}$, and noting that,

$$X^T T^*_{ii} X = X^T(S+A)X = X^T S X, \quad (13)$$

since matrix A contributes zero to the quadratic form. With this observation, condition (9) can be rewritten as requiring the matrix
$T^*_{ii} + (T^*_{ii})^T =$

$$\begin{bmatrix} 2T^*_{ii}(1,1) & T^*_{ii}(1,2)+T^*_{ii}(2,1) & \cdots \\ T^*_{ii}(1,2)+T^*_{ii}(2,1) & 2T^*_{ii}(2,2) & \\ \vdots & & \ddots \\ T^*_{ii}(1,q_i)+T^*_{ii}(q_i,1) & \cdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} T^*_{ii}(1,q_i)+T^*_{ii}(q_i,1) \\ \vdots \\ 2T^*_{ii}(q_i,q_i) \end{bmatrix}$$

$$i = 1,...,M$$

be nonnegative definite. Since a real symmetric matrix $A = [a_{ij}]$ that is diagonally dominant, *i.e.*,

$$|a_{ii}| > \sum_{j\neq i}|a_{ij}|\ \forall i,$$

and has all nonnegative diagonal elements is nonnegative definite [7]. In order to achieve global convergence of the network, it is sufficient to let

$$\begin{aligned} 2T^*_{ii}(k,k) &\geq \sum_{h\neq k,h=1}^{q_i}|T^*_{ii}(k,h)+T^*_{ii}(h,k)|, \\ i &= 1,...,M; \\ & k=1,...,q_i. \end{aligned} \quad (14)$$

Now substituting (10) into (14) yields (12). The proof is thus completed.
With the aid of Corollary 1, the connection matrix T of Example 1 can be modified to ensure global convergence of the network. From (12), the proper modifications are,

$$\begin{aligned} T_{11}(1,1) &= T_{11}(3,3)=6, T_{22}(1,1)=4, \\ T_{22}(2,2) &= 5,\ T_{22}(1,1)=7 \end{aligned}$$

Now consider the same initial state $X^0_1 = (-1,-1,\ 1)^T$ and $X^0_2 = (1,-1,\ -1)^T$ in Example 1, the following sequence of update can be obtained,

$$\begin{aligned} X'_1 &= (-1,1,1)^T; X'_2 = (1,-1,-1)^T; \\ X_1 &= (-1,1,1)^T = X'_1; \\ X_2 &= (1,-1,-1)^T = X'_2; \end{aligned}$$

It is obvious that the PSU sequence converges to the stable (fixed) point in one iteration. Moreover, state space statistics as in Example 1 were performed. It is found that all of them converge to a fixed point. No limited cycle appears.
Results in Theorem 1 also provide milder constraints on the connection matrix than those derived for the fully parallel mode case [5]. This can be demonstrated by the following example.
Example 2:
Let

$$T = \begin{bmatrix} 2 & -2 & 1 & 3 \\ 2 & 2 & 3 & 0 \\ 1 & 3 & 2 & 1 \\ 3 & 0 & -1 & 2 \end{bmatrix}$$

be the connection matrix of a Hopfield network ($N = 4$). Moreover, assume it is operated in the following fully parallel mode, *i.e.*,

$$X' = sgn\{TX\},$$

After testing the network with 16 (2^4) possible initial states, it is found that 4 of them converged to a limited cycle of period two. The corresponding matrix,

$$T^* = \begin{bmatrix} 0 & -2 & 1 & 3 \\ 2 & 0 & 3 & 0 \\ 1 & 3 & 1 & 1 \\ 3 & 0 & -1 & 1 \end{bmatrix},$$

is an indefinite matrix since the eigenvalues of $\{T^* + (T^*)^T\}$ are -4.1802, -6.1530, 6.1802, and 8.1530. Global convergence of this network is thus not guaranteed. However, if T is partitioned as in (5) and the PSU mode ((6a) & (6b)) is considered ($M = 2$, $q_1 = q_2 = 2$), we have

$$T_{11}^* = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, T_{22}^* = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

which are nonnegative and positive definite matrices, respectively. Since the conditions in Theorem 1 hold, the network is globally stable in this case. Same test as above was performed, it is found that all of them converge to a fixed point. No limited cycle appears.
From this example it is obvious that the results in Corollary 1 permit a little relaxation on the lower bound of the main diagonal elements of the connection matrix.

5. CONCLUDING REMARKS

The stability property has been studied, for Hopfield networks whose neurons are updated in PSU mode. An example was given to show that the state of Hopfield network may converge to limited cycles even if one use a connection matrix satisfying the condition derived in [3]. Then, new sufficient conditions ensuring global convergence of a Hopfield network in PSU mode are derived. They provide milder constraints on the connection matrix than those derived for the fully parallel mode case [5].

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