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ON THE POWERS OF GRAPHS WITH BOUNDED ASTEROIDAL NUMBER

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Abstract

For an undirected graph $G = (V, E)$, the k-th power G^k is the graph with the same vertex set as G such that two vertices are adjacent in G^k if and only if their distance in G is at most k. A set of vertices $A \subseteq V$ is an asteroidal set if for every vertex $a \in A$, the set $A \backslash \{a\}$ is contained in one connected component of $G - N_G[a]$, where $N_G[a]$ is the closed neighborhood of a in G. The asteroidal number of a graph G is the maximum cardinality of an asteroidal set in G. The class of graphs with asteroidal number at most s is denoted by $\mathcal{A}(s)$. In this paper, we show that if $G^k \in \mathcal{A}(s)$ for $s \geq 2$, then so is G^{k+1}. This generalizes a previous result for the family of AT-free graphs. Moreover, we consider the forbidden configurations for the powers of graphs with bounded asteroidal number. Based on these forbidden configurations, we show that every proper power of AT-free graphs is perfect.

Keywords: asteroidal triple, AT-free graphs, powers of graphs, strong perfect graph conjecture.

1. Introduction

Let $G = (V, E)$ be a graph consisting of the vertex set V and the edge set E, respectively. For a positive integer k, the kth *power* G^k is the graph with the same vertex set as G such that two vertices are adjacent in G^k if and only if their distance in G is at most k. Up to now several results concerning the family of graphs closed under power operations have been investigated. One of the first results in this field is due to Duchet [15]: If G^k is chordal (i.e., a graph with the property that every cycle of length greater than three has a chord), then so is G^{k+2}. Consequently, all odd powers of chordal graphs are chordal, whereas this is not true in general for even powers. In contrast, Bandelt et al. [1] showed that all even powers of a distance-hereditary graph (i.e., a graph G with the property that the distance of any two vertices in each connected induced subgraph equals their distance in G) are chordal.

Let $\mathcal{C}$ be a class of graphs and k a positive integer. The assertion "$G^k \in \mathcal{C} \Rightarrow G^{k+2} \in \mathcal{C}$" is therefore called the Duchet-type assertion. Besides, two analogies of Duchet-type assertion, called weakly-type assertion ($G \in \mathcal{C} \Rightarrow G^k \in \mathcal{C}$) and strongly-type assertion ($G^k \in \mathcal{C} \Rightarrow G^{k+1} \in \mathcal{C}$), are also concerned in other papers [5, 6, 13, 14, 16, 21, 24, 25]. Table 1 summarizes the existing results on the above three type assertions for various classes of graphs. For an overview of these classes of graphs, please refer to [4, 17].

Table 1: Three type assertions related to the power operations.

Classes of Graphs	Duchet-type assertion $G^k \in \mathcal{C} \Rightarrow G^{k+2} \in \mathcal{C}$	Weakly-type assertion $G \in \mathcal{C} \Rightarrow G^k \in \mathcal{C}$	Strongly-type assertion $G^k \in \mathcal{C} \Rightarrow G^{k+1} \in \mathcal{C}$
Chordal	[15]		
Strongly chordal		[13, 21]	[25]
Circular arc		[25]	
(Unit) interval			[24]
AT-free			[24]
Cocomparability		[14]	[16]
m-trapezoid			[16]
Dually chordal		[5]	
HHD-free	[6]		
Weak bipolarizable	[6]		

An independent set of a graph $G = (V, E)$ is a set of pairwise nonadjacent vertices. The *open neighborhood* $N_G(u)$ of a vertex $u \in V$ is the set $\{v \in V \mid (u,v) \in E\}$; and the *closed neighborhood* $N_G[u]$ is $N_G(u) \cup \{u\}$. For a subset $W \subset V$, we use $G - W$ to denote the subgraph of G induced by the vertex set $V \backslash W$ (i.e., $\{v \mid v \in V$ and $v \notin W\}$).

An *asteroidal triple* (AT for short) of a graph G is an independent set of three vertices such that every two vertices are joined by a path avoiding the closed neighborhood of the third. Graphs without asteroidal triples are called *asteroidal triple-free graphs* (AT-free graphs for short). Lekkerkerker and Boland [20] first introduced the concept of asteroidal triples to characterize the interval graphs. A graph is an interval graph if and only if it is chordal and AT-free. Recently, Corneil et al. [11] obtained a collection of interesting structural and algorithmic properties for AT-free graphs. Note that, the class of AT-free graphs properly contains well-known classes of graphs such as interval, permutation, trapezoid and cocomparability. Further results on AT-free graphs please refer to [7, 10, 11, 12, 19, 22].

Walter [26] generalized the concept of asteroidal triples to so-called asteroidal sets, and used asteroidal sets to characterize certain subclasses of chordal graphs. A set of vertices $A \subseteq V$ is an *asteroidal set* if for every vertex $a \in A$, the set $A \backslash \{a\}$ is contained in one connected component of $G - N_G[a]$. The *asteroidal number* of a graph G is the maximum cardinality of an asteroidal set in G. It is easy to see that each asteroidal set is an independent set and the AT-free graphs are those graphs with asteroidal number at most two. Kloks et al. [18] showed that the asteroidal number can be computed by efficient algorithms for some classes of graphs such as claw-free, HHD-free, circular-arc and circular permutation graphs, while the corresponding decision problem is NP-complete even when restricted to triangle-free 3-connected 3-regular planar graphs.

Let $s \geq 2$ be a positive integer. The class of graphs with asteroidal number at most s is denoted by $\mathcal{A}(s)$. Obviously, all the classes of graphs with bounded asteroidal number constitute a hierarchy of families by set inclusion, i.e., $\mathcal{A}(s) \subset \mathcal{A}(s+1)$. In this paper, we show that the strongly-type assertion mentioned as above holds on each class $\mathcal{A}(s)$ for $s \geq 2$ (i.e., $G^k \in \mathcal{A}(s) \Rightarrow G^{k+1} \in \mathcal{A}(s)$). This covers the previous result of [24] for the family of AT-free graphs. Besides, we consider the forbidden configurations for the powers of graphs $G \in \mathcal{A}(s)$. We show that every proper power (G^k for $k \geq 2$) of a graph $G \in \mathcal{A}(s)$ does not contain $K_{1,s+2}$, $K_{2,s+1}$ and C_{2s+1} induced subgraphs. The first two forbidden configurations generalize a partial result proposed in [14] that every proper power of a cocomparability graph has no $K_{1,4}$ and $K_{2,3}$ induced subgraphs. Moreover, based on these forbidden configurations, we show that all proper powers of AT-free graphs are perfect.

2. Graphs with bounded asteroidal number and their powers

All graphs considered in this paper are undirected, simple (i.e., without loops and multiple edges) and connected. For a graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices $u, v \in V$ is the number of edges of a shortest path from u to v in G. For convenience, we write $G^k = (V, E^k)$ where $E^k = \{(u, v) \mid u, v \in V \text{ and } d_G(u, v) \le k\}$. In particular, we call G^2 the *square* of G, and G^k for $k \ge 2$ the *proper power* of G.

A path joining two vertices u and v is termed a *u-v path.* The union of two paths P and P' with a common endvertex is denoted by $P \oplus P'$. A vertex u *misses* a path P if there are no vertices of P adjacent to u; otherwise, we say that u *intercepts* P. For any two vertices $u, v \in V$, we denote $D_G(u, v)$ as the set of vertices that intercept all u-v paths in G. Note that, a vertex $w \notin D_G(u, v)$ if and only if there exists a u-v path P in G such that w misses P. For graph-theoretic terminologies and notations not mentioned here we refer to [17].

Kloks et al. [18] show that a set A with cardinality $|A| \ge 3$ is an asteroidal set if and only if every triple of A is an AT. Based on this result, we have the following.

Lemma 2.1 *A set A with $|A| \ge 3$ is an asteroidal set of a graph G if and only if every three vertices $u, v, w \in A$, $u \notin D_G(v, w)$, $v \notin D_G(u, w)$ and $w \notin D_G(u, v)$.*

Lemma 2.2 *For every two vertices x and y of a graph $G = (V, E)$, $D_{G^k}(x, y) \subseteq D_{G^{k+1}}(x, y)$.*

Proof. Let $w \in V \backslash \{x, y\}$. We will prove that if $w \notin D_{G^{k+1}}(x, y)$ then $w \notin D_{G^k}(x, y)$. Suppose that w misses some x-y path $P = (x = u_1, u_2, \ldots, u_n = y)$ in G^{k+1}. Without loss of generality we assume that P is as short as possible. Clearly, $w \notin \{u_i \mid 1 \le i \le n\}$ and

$$d_{G^k}(w, u_i) \ge d_{G^{k+1}}(w, u_i) \ge 2 \text{ for } i = 1, \ldots, n. \quad (1)$$

Let F be the set consisting of edges of P that belong to $E^{k+1} \backslash E^k$. If F is empty, then P is still an x-y path in G^k. In this case, since w misses P in G^k, no further proof is necessary. We now consider F is nonempty as follows.

For each edge $(u_i, u_{i+1}) \in F$, let $P(u_i, u_{i+1})$ be a shortest path with length $k + 1$ joining u_i and u_{i+1} in G. Let z_i be the vertex in $P(u_i, u_{i+1})$ that is adjacent to u_i in G. It is easy to verify that z_i cannot appear in P and $(u_i, z_i), (z_i, u_{i+1}) \in E^k$. We claim that for any two edges $(u_i, u_{i+1}), (u_j, u_{j+1}) \in F$, the corresponding vertices z_i and z_j are not the same vertex of G. Suppose not, and without loss of generality assume $i < j$. Since z_i and z_j are the same vertex of G,

$$\begin{aligned} d_G(u_i, u_{j+1}) &\le d_G(u_i, z_i) + d_G(z_i, u_{j+1}) \\ &= d_G(u_i, z_i) + d_G(z_j, u_{j+1}) \\ &= 1 + k. \end{aligned}$$

This implies that $(u_i, u_{j+1}) \in E^{k+1}$. Thus, $(x = u_1, \ldots, u_i, u_{j+1}, \ldots, u_n = y)$ forms another x-y path with length less than P in G^{k+1} such that w misses the path. This leads a contradiction to the assumption.

Next, we claim that for each edge $(u_i, u_{i+1}) \in F$, the corresponding vertex z_i is not adjacent to w in G^k. It can be derived from (1) by the following implications:

$$\begin{aligned} & d_{G^{k+1}}(w, u_i) \ge 2 \\ \Rightarrow\ & d_G(w, u_i) > k + 1 \\ \Rightarrow\ & d_G(w, z_i) + d_G(z_i, u_i) > k + 1 \\ \Rightarrow\ & d_G(w, z_i) > k \\ \Rightarrow\ & d_{G^k}(w, z_i) \ge 2. \end{aligned} \quad (2)$$

Hence, for each edge $(u_i, u_{i+1}) \in F$ in P, substitute it by the edges (u_i, z_i) and (z_i, u_{i+1}). The replacements of all these edges yield another x-y

path P' with length $n-1+|F|$ in G^{k+1}. Since $(u_i, z_i), (z_i, u_{i+1}) \in E^k$, P' remains in G^k. We conclude by (1) and (2) that no vertices of P' are adjacent to w in G^k. Thus $w \notin D_{G^k}(x, y)$. This completes the proof. □

Theorem 2.3 *For any integer $s \geq 2$, $G^k \in \mathcal{A}(s)$ implies $G^{k+1} \in \mathcal{A}(s)$*

Proof. Suppose that $G^{k+1} \notin \mathcal{A}(s)$. This means that there exists an asteroidal set A with $|A| > s$ in G^{k+1}. By Lemma 2.1, every three vertices $u, v, w \in A$ satisfy $u \notin D_{G^{k+1}}(v, w)$, $v \notin D_{G^{k+1}}(u, w)$ and $w \notin D_{G^{k+1}}(u, v)$. Also, by Lemma 2.2 we know that $u \notin D_{G^k}(v, w)$, $v \notin D_{G^k}(u, w)$ and $w \notin D_{G^k}(u, v)$. So $\{u, v, w\}$ forms an AT in G^k. It follows from Lemma 2.1 that A is an asteroidal set in G^k. Thus, $G^k \notin \mathcal{A}(s)$. □

Corollary 2.4 *If G^k is AT-free, then so is G^{k+1}.*

3. Forbidden configurations

In this section, the forbidden configurations for the proper powers of graphs $G \in \mathcal{A}(s)$ are considered. For a graph $G = (V, E)$, the following lemma can be used to test that a vertex $v \in V$ misses a certain shortest path in G.

Lemma 3.1 *Let $G = (V, E)$ be a graph and $u, v, w \in V$. For any positive integer $k \geq 2$, if $d_G(u, v) \geq 2$ and $d_G(v, w) > k \geq d_G(u, w)$, then v misses every shortest u-w path in G.*

Proof. The lemma is trivial for the case $(u, w) \in E$. We now consider $d_G(u, w) \geq 2$. Let $P(u, w)$ be a shortest path joining u and w in G. Assume that there is a vertex $x \in N_G[v]$ which is contained in $P(u, w)$. Since $d_G(u, v) \geq 2$ and $d_G(v, w) > k \geq d_G(u, w)$, $x \notin \{u, v, w\}$ and $d_G(x, w) < k$. Thus, $d_G(v, w) \leq d_G(v, x) + d_G(x, w) \leq 1 + (k-1) = k$, a contradiction. □

A graph G is a cocomparability graph if its complement $\overline{G}$ is transitively orientable. Recall the previous result proposed by Damaschke [14] that every proper power of cocomparability graphs has no induced subgraphs isomorphic to $K_{1,4}$ or $K_{2,3}$. Since the cocomparability graphs are properly contained in the class of AT-free graphs, the following two lemmas generalize the result of [14].

Lemma 3.2 *Every proper power G^k of a graph $G \in \mathcal{A}(s)$ for $s \geq 2$ does not contain $K_{1,s+2}$ as an induced subgraph.*

Proof. Suppose to the contrary. Let $W = \{x, y_1, y_2, \ldots, y_{s+1}, y_{s+2}\}$ be a set of vertices that induces a $K_{1,s+2}$ in G^k where y_i for $i = 1, \ldots, s+2$ are independent vertices. Then, for $i, j \in \{1, \ldots, s+2\}$ with $i \neq j$, $d_G(y_i, y_j) > k \geq d_G(y_i, x)$. Because the subgraph induced by W in G^k certainly does not include any two adjacent edges belonging to E, assume without loss of generality that y_{s+2} is the only vertex which may be adjacent to x in G, and $d_G(y_i, x) \geq 2$ for $i = 1, \ldots, s+1$. Let $P(y_i, x)$ be a shortest path joining y_i and x in G and let $Y = W \backslash \{x, y_{s+2}\}$. We now show that Y forms an asteroidal set of $s+1$ vertices in G. For any two vertices $y_i, y_j \in Y$, since $d_G(x, y_i) \geq 2$ and $d_G(y_i, y_j) > k \geq d_G(x, y_j)$, it follows from Lemma 3.1 that y_i misses $P(x, y_j)$ in G. Thus, for every three vertices $y_h, y_i, y_j \in Y$ in G, we conclude that y_i misses $P(y_h, x) \oplus P(x, y_j)$, y_j misses $P(y_h, x) \oplus P(x, y_i)$, and y_h misses $P(y_i, x) \oplus P(x, y_j)$. By Lemma 2.1, Y is an asteroidal set of $s+1$ vertices in G. Thus $G \notin \mathcal{A}(s)$ and the lemma follows. □

Lemma 3.3 *Every proper power G^k of a graph $G \in \mathcal{A}(s)$ for $s \geq 2$ does not contain $K_{2,s+1}$ as an induced subgraph.*

Proof. Suppose to the contrary. Let $X \cup Y$ be a set of vertices that induces a $K_{2,s+1}$ in G^k where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{s+1}\}$ are two independent sets. Then $d_G(x_i, y_j) \leq k$ for $i \in \{1, 2\}$ and $j \in \{1, \ldots, s+1\}$, $d_G(x_1, x_2) > k$, and $d_G(y_i, y_j) > k$ for $i, j \in \{1, \ldots, s+1\}$ with $i \neq j$. We first consider the subgraph $K_{1,s+1}$ induced by the vertex set $\{x_1\} \cup Y$ in

G^k. By the same argument as the proof of Lemma 3.2, we may assume that $d_G(y_i, x_1) \geq 2$ for $i = 1, \ldots, s$. Let $Y' = Y \backslash \{y_{s+1}\}$. Consequently, Y' forms an asteroidal set of s vertices in G, and for every two vertices $y_i, y_j \in Y'$ with $i \neq j$, $y_i \notin D_G(y_j, x_1)$ and $y_j \notin D_G(y_i, x_1)$. The aim of the following proof is to show that $\{x_1\} \cup Y'$ is an asteroidal set of $s+1$ vertices in G. Hence, we only need to show that $x_1 \notin D_G(y_i, y_j)$ for $y_i, y_j \in Y'$.

Let $P(y_i, x_2)$ be a shortest path joining y_i and x_2 in G for $1 \leq i \leq s$. For any vertex $y_i \in Y'$, since $d_G(y_i, x_1) \geq 2$ and $d_G(x_1, x_2) > k \geq d_G(y_i, x_2)$, it follows from Lemma 3.1 that x_1 misses $P(y_i, x_2)$ in G. So that for any two vertices $y_i, y_j \in Y'$, x_1 misses $P(y_i, x_2) \oplus P(x_2, y_j)$ in G. Thus $x_1 \notin D_G(y_i, y_j)$. Since $\{x_1\} \cup Y'$ forms an asteroidal set of $s+1$ vertices in G, it contradicts that $G \in \mathcal{A}(s)$ and the lemma follows. □

Note that if a graph G is C_n-free (no chordless cycle of length n), it does not imply that every proper power of G remains C_n-free. For example, the square of a chordal graph is not necessarily chordal. Since every cycle C_{2s+2} for $s \geq 2$ contains an asteroidal set of size $s+1$, a graph $G \in \mathcal{A}(s)$ has no such a cycle. In the following, we will show that every proper power G^k of a graph $G \in \mathcal{A}(s)$ contains no chordless cycle of length $2s+1$.

Lemma 3.4 *Every proper power G^k of a graph $G \in \mathcal{A}(s)$ for $s \geq 2$ does not contain C_{2s+1} as an induced subgraph.*

Proof. Suppose to the contrary. Let $C = (u_0, u_1, \ldots, u_{2s}, u_0)$ be a chordless cycle in G^k for some $k \geq 2$. Clearly, $d_G(u_i, u_j) \leq k$ for $|i - j| = 1$ and $d_G(u_i, u_j) > k$ for $|i - j| \geq 2$ (where the indices of the terms u_i and u_j are always taken modulo $2s+1$). Since C does not include any two consecutive edges belonging to E, without loss of generality we assume that $d_G(u_0, u_1) \geq 2$. Let $W = \{u_1, u_3, u_5, \cdots, u_{2s-1}, u_0\}$. We will show that W forms an asteroidal set of $s+1$ vertices in G. For any three distinct vertices $u_h, u_i, u_j \in W$, let $P_C(u_i, u_j, u_h)$ denote the path joining u_i and u_j and avoiding u_h in C. Due to the induced property of C, it follows that for every three vertices $u_h, u_i, u_j \in W$ in G^k with $|h-i| \geq 2$, $|i-j| \geq 2$, and $|h-j| \geq 2$, u_h misses $P_C(u_i, u_j, u_h)$, u_i misses $P_C(u_j, u_h, u_i)$, and u_j misses $P_C(u_h, u_i, u_j)$. Thus, $\{u_h, u_i, u_j\}$ forms an AT in G^k. By Lemma 2.2, $\{u_h, u_i, u_j\}$ is also an AT in G. To complete the proof, we only need to verify that for every vertex $u_h \in W \backslash \{u_0, u_1\}$, the set $\{u_0, u_1, u_h\}$ constitutes an AT in G.

Let $P(u_i, u_j)$ be a shortest path joining u_i and u_j in G for $i \neq j$. For every vertex $u_h \in W \backslash \{u_0, u_1\}$, since $d_G(u_0, u_h) > k$ and $d_G(u_h, u_1) > k \geq d_G(u_0, u_1)$, it follows from Lemma 3.1 that u_h misses $P(u_0, u_1)$ in G. Using the same argument as above, a similar proof shows that u_0 misses $P(u_{i-1}, u_i)$ for $i = 3, \ldots, h$, and u_1 misses $P(u_j, u_{j+1})$ for $j = h, \ldots, 2s-1$, respectively, in G. In particular, since $d_G(u_0, u_1) \geq 2$, u_0 misses $P(u_1, u_2)$, and u_1 misses $P(u_{2s}, u_0)$ in G. Therefore, u_0 misses the path $P(u_1, u_2) \oplus P(u_2, u_3) \oplus \cdots \oplus P(u_{h-1}, u_h)$, and u_1 misses the path $P(u_h, u_{h+1}) \oplus P(u_{h+1}, u_{h+2}) \oplus \cdots \oplus P(u_{2s}, u_0)$. Consequently, $\{u_0, u_1, u_h\}$ forms an AT in G. This completes the proof. □

We summary the above results as the following theorem.

Theorem 3.5 *Let G be a graph in the class $\mathcal{A}(s)$ for $s \geq 2$. Then, every proper power G^k has no $K_{1,s+2}$, $K_{2,s+1}$ and C_{2s+1} induced subgraphs.*

4. Perfection on powers of AT-free graphs

A graph G is called *perfect* if for every induced subgraph H of G, the chromatic number of H (i.e., the minimum number of colours is necessary to colour the vertices of H such that any two adjacent vertices have different colours) equals the largest size of a clique of H. For background results on perfect graphs, see [3, 17]. A *hole* is a chordless cycle of length at least four, while an *antihole* is the complement of a hole. We say that G is a *Berge graph* if it contains no odd hole and no odd antihole. Berge [2] conjectured that a graph is perfect if and only if it is a Berge graph. The "only if" part of this conjecture is easy to check. However, the converse remains open today and is referred to as the Strong Perfect Graph Conjecture (abbreviated SPGC). In the past few years, the SPGC has been proved to be valid by restricting it to certain special classes of graphs defined by forbidden induced subgraphs. For example, see [23] for the case of the claw-free Berge graphs (a Berge graph has no $K_{1,3}$ induced subgraph).

For a set S of vertices in a graph G, S is called a *star-cutset* if $G - S$ is disconnected and there exists a vertex $u \in S$ such that $S \subseteq N_G[u]$. A graph is *minimal imperfect* if it is not perfect and all of its proper induced subgraphs are perfect. The structure of minimal imperfect graphs has been intensively studied. One of the most useful properties of minimal imperfect graphs is the Star Cutset Lemma which was proposed by Chvátal [9]. This lemma states that no minimal imperfect graph contains a star-cutset and is very useful for proving theorems on perfect graphs.

Because the odd antihole with at least five vertices is AT-free, the AT-free graphs need not be perfect. Let $\mathcal{P}(\mathcal{A}(s))$ denote the class of graphs containing all proper powers of graphs $G \in \mathcal{A}(s)$ for $s \geq 2$. In the remainder, we will show that every graph $G \in \mathcal{P}(\mathcal{A}(2))$ has no induced odd antihole. Using this property together with the forbidden configurations given in the previous section, we will show that $\mathcal{P}(\mathcal{A}(2))$ is properly contained in the class of perfect graphs.

Lemma 4.1 *Let $G = (V, E)$ be an AT-free graph and $n \geq 5$ be an odd integer. Then, every proper power G^k does not contain $\overline{C}_n$ as an induced subgraph.*

Proof. For the case $n = 5$, since C_5 is isomorphic to its complement $\overline{C}_5$, the result follows directly from Lemma 3.3 that every proper power of AT-free graphs does not contain $\overline{C}_5$ as an induced subgraph. For odd integer $n \geq 7$, we suppose to the contrary. Let $(u_0, u_1, \ldots, u_{n-1}, u_0)$ be a chordless cycle in the complement of G^k, and let H be the subgraph of G^k induced by the vertices u_i for $i = 0, \ldots, n-1$. That is, H is isomorphic to $\overline{C}_n$ in which any two vertices u_i and u_j have the distance $d_G(u_i, u_j) \leq k$ for $|i - j| \geq 2$ or $d_G(u_i, u_j) > k$ for $|i - j| = 1$ (where the indices of the terms u_i and u_j are always taken modulo n). Let $P(u_i, u_j)$ denote a shortest path joining u_i and u_j in G for $i \neq j$. We consider the following two cases.

Case 1: at least one of edges (u_0, u_2) and (u_0, u_{n-2}) in H does not belong to E. Without loss of generality we assume that $2 \leq d_G(u_0, u_2) \leq k$. Clearly, the set $\{u_0, u_1, u_2\}$ is an independent set in G. Since $d_G(u_0, u_1) > k$ and $d_G(u_1, u_2) > k \geq d_G(u_0, u_2)$, it follows from Lemma 3.1 that u_1 misses $P(u_0, u_2)$ in G. Similarly, we can show that u_2 misses $P(u_1, u_3)$ in G. Also, since $d_G(u_0, u_2) \geq 2$ and $d_G(u_2, u_3) > k \geq d_G(u_0, u_3)$, it follows from Lemma 3.1 that u_2 misses $P(u_0, u_3)$ in G. Thus, u_2 misses $P(u_0, u_3) \oplus P(u_3, u_1)$ in G. By the symmetric property of H, u_0 misses $P(u_2, u_{n-1}) \oplus P(u_{n-1}, u_1)$ in G. So $\{u_0, u_1, u_2\}$ forms an AT in G, a contradiction.

Case 2: both the edges (u_0, u_2) and (u_0, u_{n-2}) in H belong to E. Since the degree of u_0 in H is even and no two vertices u_i and u_{i+1} $(i = 2, \ldots, n-3)$ are adjacent to u_0 in G, an easy argument shows that there exist at least two vertices u_m and u_{m+1} for some $m = 3, \ldots, n-4$ such that $(u_0, u_m), (u_0, u_{m+1}) \notin E$; i.e., $2 \leq d_G(u_0, u_m), d_G(u_0, u_{m+1}) \leq k$. Clearly, the set $\{u_0, u_m, u_{m+1}\}$ is an independent set in G. Since $d_G(u_0, u_m) \geq 2$ and $d_G(u_m, u_{m+1}) > k \geq d_G(u_0, u_{m+1})$, it follows from Lemma 3.1 that u_m misses $P(u_0, u_{m+1})$ in G. Similarly, we can show that u_{m+1} misses $P(u_0, u_m)$ in G. Also, since $3 \leq m \leq n-4$, $d_G(u_m, u_1) \leq k$ and $d_G(u_{m+1}, u_1) \leq k$. Thus, for $j \in \{m, m+1\}$, we have $d_G(u_j, u_0) \geq 2$ and $d_G(u_0, u_1) > k \geq d_G(u_j, u_1)$. This follows from Lemma 3.1 that u_0 misses $P(u_m, u_1) \oplus P(u_1, u_{m+1})$ in G. So $\{u_0, u_m, u_{m+1}\}$ forms an AT in G, a contradiction. □

Corollary 4.2 *Every proper power of AT-free graphs is a Berge graph.*

Proof. Corollary 2.4 shows that every proper power of AT-free graphs remains AT-free. Since every chordless n-cycle for $n \geq 6$ contains an AT, AT-free graphs have no such a cycle. Thus, the result follows directly from Lemma 3.4 and Lemma 4.1. □

Theorem 4.3 *Every proper power of AT-free graphs is perfect.*

Proof. Suppose to the contrary. Let $G = (V, E)$ be an AT-free graph, and assume that there is a proper power G^k which is not perfect. Then, G^k has an induced subgraph H that is minimal imperfect (in fact, every graph with no more than four vertices is perfect). By Corollary 4.2, G^k is Berge and so is H. Since claw-free Berge graphs are perfect [23], it follows that H contains a claw. Let u, x, y, z induce a claw in H with edges $(u, x), (u, y)$ and (u, z). Since the set $N_H[u] \backslash \{x, y\}$ is not a star-cutset in H, it follows that there is a chordless path $P(x, y)$ joining x and y in H whose inner vertices are all nonadjacent to u. Since H contains no hole with at least five vertices, $P(x, y)$ has exactly two edges.

Let z' be the vertex of $P(x, y)$ joining x and y. By Lemma 3.3, G^k is $K_{2,3}$-free and so is H. Thus, z and z' are nonadjacent in H. A similar argument shows that there exist two chordless paths $P(y, z) = (y, x', z)$ and $P(x, z) = (x, y', z)$ in H such that x' is adjacent to neither u nor x, and y' is adjacent to neither u nor y. It is easy to see that $(x', y') \notin E^k$; otherwise, x, u, y, x', y' induce a C_5 in G^k. By the same argument as above, $(y', z') \notin E^k$ and $(x', z') \notin E^k$. Therefore, x, y, z, x', y', z' induce a C_6 in G^k. However, this is impossible for AT-free graphs. □

5. Conclusion

In this paper, we show that the class of graphs with bounded asteroidal number is closed under power operations. In addition, we show that a graph in $\mathcal{P}(\mathcal{A}(s))$ for $s \geq 2$ does not contain $K_{1,s+2}$, $K_{2,s+1}$ and C_{2s+1} as induced subgraphs. In particular, we show that every proper power of AT-free graphs has no induced odd antihole. Consequently, $\mathcal{P}(\mathcal{A}(2))$ is contained in the class of perfect AT-free graphs. Since $K_{1,4}$ is a perfect AT-free graph and every graph in $\mathcal{P}(\mathcal{A}(2))$ has no $K_{1,4}$ induced subgraph, we conclude that $\mathcal{P}(\mathcal{A}(2))$ is strictly contained in the class of perfect AT-free graphs.

A previous result proposed in [8] showed that the class of cocomparability graphs is also strictly contained in the class of perfect AT-free graphs. Since the class of cocomparability graphs is the largest well-known subclass of perfect AT-free graphs, a natural question to ask is whether every graph in $\mathcal{P}(\mathcal{A}(2))$ is a cocomparability graph.

References

[1] H.-J. Bandelt, A. Henkmann and F. Nicolai, Powers of distance-hereditary graphs, *Discrete Mathematics* 145 (1995) 37-60.

[2] C. Berge, Les problèmes de coloration en theéorie des graphes, Publ. Inst. Statnt. Univ. Paris 9 (1960) 123-160.

[3] C. Berge and V. Chvátal (eds.), Topics on Perfect Graphs (North-Holland, Amsterdam, 1984).

[4] A. Brandstädt, Special graph classes — a survey, Schriftenreihe des Fachbereichs Mathematik, SM-DU-199, Universität Duisburg, 1991.

[5] A. Brandstädt, F. F. Dragan, V. D. Chepoi and V. I. Voloshin, Dually chordal graphs, in: Proc. of WG'93 Conference, Lecture Notes in Computer Science, Springer-Verlag, Vol. 790 (1994) 237-251.

[6] A. Brandstädt, V. B. Le and T. Szymczak, Duchet-type theorems for powers of HHD-free graphs, *Discrete Mathematics* 177 (1997) 9-16.

[7] H. Broersma, T. Kloks, D. Kratsch and H. Muller, Independent sets in asteroidal triple-free graphs, in: Proc. of ICALP'97 Conference, Lecture Notes in Computer Science, Springer-Verlag, Vol. 1256 (1997) 760-770.

[8] F. Cheah, A recognition algorithm for II-graphs, Doctoral thesis, Department of Computer Science, University of Toronto, 1990.

[9] V. Chvátal, Star-cutsets and perfect graphs, *Journal of Combinatorial Theory* (B) 39 (1985) 189-199.

[10] D. G. Corneil, S. Olariu and L. Stewart, A linear time algorithm to compute a dominating path in an AT-free graph, *Information Processing Letters* 54 (1995) 253-257.

[11] D. G. Corneil, S. Olariu and L. Stewart, Asteroidal triple-free graphs, *SIAM Journal on Discrete Mathematics* 10 (1997) 399-430.

[12] D. G. Corneil, S. Olariu and L. Stewart, Linear time algorithms for dominating pairs in asteroidal triple-free graphs, in: Proc. of ICALP'95 Conference, Lecture Notes in Computer Science, Springer-Verlag, Vol. 944 (1995) 292-302.

[13] E. Dahlhaus and P. Duchet, On strongly chordal graphs, *Ars Combinatoria* 24B (1987) 23-30.

[14] P. Damaschke, Distances in cocomparability graphs and their powers, *Discrete Applied Mathematics* 35 (1992) 67-72.

[15] P. Duchet, Classical perfect graphs, *Annals of Discrete Mathematics* 21 (1984) 67-96.

[16] C. Flotow, On powers of m-trapezoid graphs, *Discrete Applied Mathematics* 63 (1995) 187-192.

[17] M. C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).

[18] T. Kloks, D. Kratsch and H. Müller, Asteroidal sets in graph, in: Proc. of WG'97 Conference, Lecture Notes in Computer Science, Springer-Verlag, Vol. 1335 (1997) 229-241.

[19] T. Kloks, D. Kratsch and J. Spinrad, On treewidth and minimum fill-in of asteroidal triple-free graphs, *Theoretical Computer Science* 175 (1997) 309-335.

[20] C. G. Lekkerkerker and J. C. Boland, Representation of a finite graph by a set of intervals on the real line, *Fundamenta Mathematicae* 51 (1962) 45-64.

[21] A. Lubiw, Γ-free matrices, M. S. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1982.

[22] R. H. Möhring, Triangulating graphs without asteroidal triples, *Discrete Applied Mathematics* 64 (1996) 281-287.

[23] K. R. Parthasarathy and G. Ravindra, The strong perfect graph conjecture is true for $K_{1,3}$-free graphs, *Journal of Combinatorial Theory* (B) 28 (1976) 212-223.

[24] A. Raychaudhuri, On powers of interval and unit interval graphs, *Congressus Numerantium* 59 (1987) 235-242.

[25] A. Raychaudhuri, On powers of strongly chordal and circular arc graphs, *Ars Combinatoria* 34 (1992) 147-160.

[26] J. R. Walter, Representations of chordal graphs as subtrees of a tree, *Journal of Graph Theory* 2 (1978) 265-267.