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Node-to-Set Cluster Fault Tolerant Routing in Star Networks *

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Abstract

Consider the following node-to-set routing problem in the n-dimensional star graph G_n: given a node s and a set of nodes $T = \{t_1, \ldots, t_k\}$, $2 \leq k \leq n-1$, find k node-disjoint paths $s \rightarrow t_i$, $1 \leq i \leq k$. From Menger's theorem, it is known that G_n can tolerate at most $(n-1) - k$ arbitrary faulty nodes for node-to-set routing problem. In this paper, we prove that G_n can tolerate as many as $(n-1) - k$ arbitrary faulty clusters of diameter at most 2, where the faulty cluster is a connected subgraph of G_n such that all its nodes are faulty. This result implies that G_n can tolerate as many as $n(n-1-k)$ faulty nodes for node-to-set routing if the faulty nodes can be covered by $(n-1)-k$ subgraphs of diameter at most 2. We also show an algorithm which, in the presence of up to $(n-1)-k$ faulty clusters of diameter at most 2, finds the k routing paths of length at most $d(G_n)+9$ for node-to-set routing in $O(|F|+kn)$ time, where $d(G_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ is the diameter of G_n and $|F|$ is the total number of faulty nodes in faulty clusters.

1 Introduction

Node fault tolerant routing is one of the central issues in today's interconnection networks and it has been discussed extensively [11, 10, 2]. Most work for node fault tolerant routing has been done on a graph G when a certain number of *arbitrary nodes* failed. For a specific routing problem, we say a graph G can tolerate l faulty nodes if given at most l arbitrary faulty nodes, the required routing paths exist for the routing problem. In the presence of more than l faulty nodes, if the routing paths do not always exist, what are the sufficient conditions for the existence of the routing path? This is one of the fundamental problems in node fault tolerant routing and several approaches such as *forbidden faulty set* and *cluster fault tolerant routing* have been developed [14, 12, 7, 9]. The approach of *cluster fault tolerant routing* (abbreviated as CFT routing in what follows) is to reduce the number of "faulty nodes" that the routing has to deal with using subgraphs of small diameters to cover the faulty nodes [7, 9]. In CFT routing, a connected subgraph is called a *cluster* and a cluster is called *faulty cluster* if all its nodes are faulty. It is previously known that for several routing problems in some interconnection networks with regular structures, if multiple faulty nodes can be covered by a cluster of small diameter, those faulty nodes can be viewed as one "faulty node" rather than several arbitrary ones [7, 9]. In practice, failed processors can often be covered by fewer clusters of small diameters. For example, two routing jobs, one is between nodes s_1 and t_1 and the other is between s_2 and t_2, are performed simultaneously. The nodes in the routing path between s_1 and t_1 may be viewed as faulty nodes by the routing between s_2 and t_2, and vice versa. In the above example, m faulty nodes can be covered by $\lceil m/d \rceil$ clusters of diameter d.

Star graphs are an attractive alternative to the hypercubes. Like hypercubes, star graphs posses rich recursive structure and symmetry properties. In addition, with regard to the important properties of node degree and diameter, star graphs are shown to be markedly superior. A number of efficient algorithms on star graphs, which exploit its versatility, have been reported [1, 4, 6, 5] Further efforts for node-disjoint paths (abbreviated as disjoint paths in what follows) and fault tolerant routing problems in star graphs have also been made by several researchers [4, 15, 3, 13, 16, 6] M. Dietzfelbinger et. al. [4] first studied the following disjoint paths problems in star graphs: finding disjoint paths between two nodes s and t (node-to-node) and finding disjoint paths between a node s and a set of nodes $T = \{t_1, \ldots, t_k\}$ (node-to-set). Node-to-node disjoint paths of optimal length in star graphs were found in [15, 3, 13, 16]. Node-to-node fault tolerant routing and node-to-node CFT routing in star graphs were discussed in [8, 9]. In this paper, we continue the efforts in the same direction and study node-to-set CFT routing in the n-dimensional star graphs G_n:

- Given a set **F** of faulty clusters, a non-faulty node s, and a set of non-faulty nodes $T = \{t_1, \ldots, t_k\}$, $2 \leq k \leq n-1$, in G_n, find k fault-free disjoint paths $s \rightarrow t_i$, $2 \leq i \leq k$.

If $\mathbf{F} = \emptyset$ then the above problem becomes node-to-set disjoint paths problem. For node-to-set disjoint paths problem, from Menger's theorem, k disjoint paths exist if $k \leq n-1$. For node-to-set disjoint paths problem

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in G_n, $n-1$ disjoint paths of length at most $d(G_n)+4$ can be found in $O(n^2)$ time, where $d(G_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ is the diameter of G_n [6].

In this paper, we show that G_n can tolerate as many as $(n-1)-k$ faulty clusters of diameter at most 2 for node-to-set CFT routing. In particular, we give an algorithm which, given at most $(n-1)-k$ faulty clusters of diameter at most 2 in G_n, constructs k fault-free disjoint paths of length at most $d(G_n)+9$ for node-to-set CFT routing in $O(|F|+kn)$ time, where $|F|$ is the total number of faulty nodes in faulty clusters. Since a cluster of diameter 2 in G_n has as many as n nodes, the above result implies that G_n can tolerate as many as $n(n-1-k)$ faulty nodes for node-to-set CFT routing if the faulty nodes can be covered by certain clusters. This is in contrasts to the previous result that G_n can tolerate at most $(n-1)-k$ arbitrary faulty nodes for node-to-set routing.

The rest of this paper is organized as follows. Section 2 gives the preliminaries of this paper. Node-to-set CFT routing is discussed in Section 3 and some conclusional remarks are given in the final section.

2 Preliminaries

Graphs considered in this paper are undirected graphs. A path in graph is a sequence of edges of the form $(s_1,s_2)(s_2,s_3)\dots(s_{k-1},s_k)$. The length of a path is the number of edges in the path. We sometimes denote the path from s_1 to s_k by $s_1 \rightarrow s_k$. For a path $P = (s_1,s_2)(s_2,s_3)\dots(s_{k-1},s_k)$, P is also used to denote the set of nodes $\{s_1,\dots,s_k\}$ that appear in path P, if there is no confusion arises. For any two nodes $s,t \in G$, $d(s,t)$ denotes the distance between s and t, i.e., the length of the shortest path connecting s and t. The *diameter* of G is $d(G) = \max\{d(s,t)|s,t \in G\}$. The *eccentricity* of $s \in G$ is $e(s) = \max\{d(s,t)|t \in G\}$ and the *radius* of G is $r(G) = \min\{e(s)|s \in G\}$. The *center node* of a graph G is the node $s \in G$ such that $e(s) \leq r(G)$. Graph G is connected if there exists a path between any two of its nodes and G is disconnected otherwise. The connectivity of the graph G is defined to be the minimum number of nodes whose removal disconnects G or reduces it to a single node. Graph G is k-connected if its connectivity is k.

Let $\mathbf{F}$ be a set of faulty clusters in a graph G. $|\mathbf{F}|$ denotes the cardinality of $\mathbf{F}$, $d(\mathbf{F}) = \max\{d(C)|C \in \mathbf{F}\}$ denotes the diameter of $\mathbf{F}$, and $F = \bigcup_{C\in\mathbf{F}} C$ denotes the set of nodes of the clusters in $\mathbf{F}$. For a particular CFT routing problem in a graph G, a triple (m,d,l) is called a *features number* of G, if for any set $\mathbf{F}$ of faulty clusters in G with $|\mathbf{F}| \leq m$, $d(\mathbf{F}) \leq d$, and $|F| \leq l$, the required routing paths exist for the routing problem. A features number (m,d,l) is called an *optimum features number* (denoted as OCFT number) for a specific CFT routing problem, if for any $(m,d,l) < (m',d',l')$, (m',d',l') is not a features number for the problem.[1] For a graph G and a triple (m,d,l) with $l = m \times \max\{|C||C \subseteq G,\ d(C) = d\}$, the triple will be denoted (and only in this case) by only two parameters (m,d) in the rest of the paper. A pair (m,d) is an OCFT number if (m,d) is a features number and for any (m',d') with $(m,d) < (m',d')$, (m',d') is not a features number.

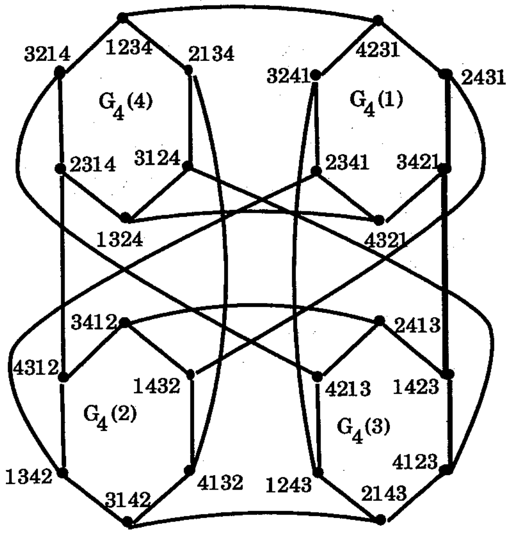

Figure 1: A 4-dimensional star graph.

An n-dimensional star graph is a graph G_n, where the nodes of G_n are in a 1-1 correspondence with the permutations $[p_1,p_2,\dots,p_n]$ of the set $<n> = \{1,2,\dots,n\}$. Two nodes of G_n are connected by an edge if and only if the permutation of one node can be obtained from the other by interchanging the first symbol p_1 with the ith symbol p_i, $2 \leq i \leq n$. This interchange of the symbols in position 1 with position i is called a *transposition*. For node $s = [p_1,p_2,\dots,p_n]$, $s^{(i)}$ denotes the node $[p_i,p_2,\dots p_{i-1},p_1,p_{i+1},\dots,p_n]$, obtained by transposition i on s. Similarly, $s^{(i_1,i_2,\dots,i_k)}$ denotes the node obtained after performing k transpositions ($s^{(1)} = s^{(i,i)} = s$). Figure 1 gives an G_4. G_n has $n!$ nodes, and $n! \times \frac{(n-1)}{2}$ edges. It has uniform node degree $n-1$ and diameter $d(G_n) = \lfloor \frac{3(n-1)}{2} \rfloor$. G_n is node and edge symmetric and is $(n-1)$-connected. Star graphs have a highly recursive structure. G_n is made of n copies of G_{n-1}. Consider the partition of nodes of G_n into n mutually disjoint subsets $S_n(k) = \{[p_1,p_2,\dots,p_{n-1},k]|p_j \in <n> -\{k\}$ for $j \neq n, p_j \neq p_l$ and $j \neq l\}$, $1 \leq k \leq n$, where $<n> -\{k\} = \{i|i \in <n>$ and $i \notin \{k\}\}$. In G_n, the induced subgraphs of the set $S_n(k)$, $1 \leq k \leq n$, is each an $(n-1)$-dimensional star graph denoted as $G_n(k)$.

We prove in this paper that $(n-1-k,2)$ is an OCFT number of G_n for node-to-set CFT routing.

[1] The partial order $\leq$ on (m,d,l) is defined as: $(m,d,l) \leq (m',d',l')$ if $m \leq m'$, $d \leq d'$, and $l \leq l'$ and $(m,d,l) < (m',d',l')$ if $(m,d,l) \leq (m',d',l')$ and $(m,d,l) \neq (m',d',l')$.

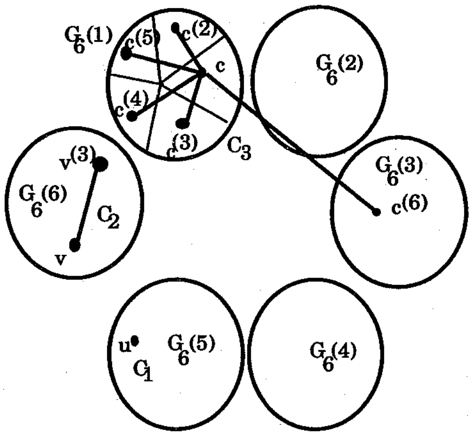

Figure 2: Example 1.

3 Node-to-Set CFT Routing

We first show some topological properties of star graphs. For a node $s = [p_1, p_2, \ldots, p_{n-1}k] \in G_n(k)$, we have $s^{(n)} \in G_n(p_1)$. We will call the node s a *port* to $G_n(p_1)$. Each node $s \in G_n(k)$ is a port to exactly one substar $G_n(p_1)$. The set of nodes

$$C_n(k,l) = \{s | s \in G_n(k) \text{ and } s^{(n)} \in G_n(l)\}$$

is called the *port-set* of $G_n(k)$ to $G_n(l)$. Clearly, the nodes in $G_n(k)$ can be partitioned into $n-1$ port-sets $C_n(k,l)$, $l \in <n> -\{k\}$, with $|C_n(k,l)| = (n-2)!$ and $C_n(k,l) \bigcap C_n(k,m) = \emptyset$ for $l \neq m$. For $s = [p_1, p_2, \ldots, p_n] \in G_n(p_n)$, $s \in C_n(p_n, p_1)$ and its neighbor nodes $s^{(i)} = [p_i, p_2, \ldots, p_{i-1}, p_1, p_{i+1}, \ldots, p_n] \in C_n(p_n, p_i)$, $i \in <n> -\{1, n\}$. Given a cluster $C = \{u = [p_1, \ldots, p_n], u^{(i)}, 2 \leq i \leq n\}$ of G_n, the nodes of C appears in two substars of G_n, with u and $u^{(i)}$, $2 \leq i \leq n-1$ in $G_n(p_n)$ and $c^{(n)}$ in $G_n(p_1)$. From this, the nodes of a cluster of diameter at most 2 appear in at most two substars of G_n.

Example 1: $C_1 = \{u = (2,3,1,4,6,5)\}$, a cluster of diameter 0; $C_2 = \{v = (4,3,2,1,5,6), v^{(3)} = (2,3,4,1,5,6)\}$, a cluster of diameter 1; and $C_3 = \{c = (3,2,4,5,6,1), c^{(i)}, 2 \leq i \leq 6\}$, a cluster of diameter 2 (see Figure 2).

The nodes of cluster C_3 appear in two substars of G_6, $c, c^{(i)}, 2 \leq i \leq 5$ appear in $G_6(1)$, while $c^{(6)}$ appears in $G_6(3)$. The node c is the center node of C_3.

Lemma 1 *There are no odd length cycle and no cycle of length 4 in G_n.*

Proof: For any node s, clearly $s \to s^{(i)} \to s^{(i,j)} \to s^{(i,j,i)} \to s^{(i,j,i,j)} \to s^{(i,j,i,j,i)} \to s^{(i,j,i,j,i,j)} = s$, where $2 \leq i \neq j \leq n$, is the shortest non-trivial path from s to s itself. □

Lemma 2 *For distinct nodes s_1 and s_2 in the same port-set of a substar, $dis(s_1, s_2) \geq 3$.*

Proof: Assume $s_1 = [p_1, p_2, \ldots, p_n]$ and $s_2 = [q_1, q_2, \ldots, q_n]$ are distinct nodes in the same port-set $C_n(p_n, p_1)$ of the substar $G_n(p_n)$. Then, $p_1 = q_1$ and $p_n = q_n$. Since $s_1 \neq s_2$ and s_1 and s_2 are permutations of $<n>$, there exist at least two positions i and j, $2 \leq i \neq j \leq n-1$, such that $p_i \neq q_i$ and $p_j \neq q_j$. Therefore, $dis(s_1, s_2) \geq 3$. □

The following results for node-to-node CFT routing in G_n will be used in node-to-set CFT routing.

Lemma 3 *[9] Given a set F, $|F| \leq n-2$, of faulty nodes, and non-faulty nodes s and t in G_n, a fault-free path of length at most $d(G_n)+3$ can be found in $O(n)$ time.*

Lemma 4 *[9] Given a set $\mathbf{F}$ of faulty clusters with $|\mathbf{F}| \leq n-2$ and $d(\mathbf{F}) \leq 2$, and non-faulty nodes s and t in G_n, a fault-free path of length at most $d(G_n)+8$ can be found in $O(|F|+n)$ time, where $F = \bigcup_{C \in \mathbf{F}} C$.*

The outline of the algorithm for node-to-set CFT routing in G_n is as follows. Given a set $\mathbf{F}$ of faulty clusters, non-faulty nodes $s = [p_1, \ldots, p_n]$ and $T = \{t_1, t_2, \ldots, t_k\}$ in G_n, the algorithm first find k destination substars $G_n(l_i)$ such that $G_n(l_i)$ does not have any center node of faulty clusters and there is a fault-free path $s \to g_{l_i} \in G_n(l_i)$ of length at most 2. Next the nodes of T are routed into destination substars, one substar for one node, by fault-free disjoint paths $t_i \to h_{l_i} \in G_n(l_i)$ of length at most 3. Finally, h_{l_i} is connected to g_{l_i} by a fault-free path in $G_n(l_i)$. We first prove that destination substars can be found.

Lemma 5 *Given a set $\mathbf{F}$ of faulty clusters with $|\mathbf{F}| \leq n-1-k$ and $d(\mathbf{F}) \leq 2$, and a non-faulty node $s = [p_1, \ldots, p_n]$ in G_n, k substars $G_n(l_i)$, $l_i \in <n> -\{p_n\}$, can be found such that $G_n(l_i)$ does not have any center node of faulty clusters and there is a fault-free path $s \to g_{l_i} \in G_n(l_i)$ of length at most 2.*

Proof: For $s = [p_1, p_2, \ldots, p_n]$, its neighbor node $s^{(i)}$, $2 \leq i \leq n-1$, is a port to the substar $G_n(p_i)$, and path $s \to s^{(i)} \to s^{(i,n)} = g_{p_i} \in G_n(p_i)$ is a path of length 2. $s \to s^{(n)} = g_{p_n} \in G_n(p_n)$ is a path of length 1. We now prove that any faulty cluster C with $d(C) \leq 2$ and $s \notin C$ can block at most one of the above $n-1$ paths $s \to g_{p_i}$, $2 \leq i \leq n$. From Lemma 1 and $d(C) \leq 2$, if $s^{(i)} \in C$, $s^{(i)}$ is any neighbor node of s, then $s^{(j)} \notin C$ and $s^{(j,n)} \notin C$ for $2 \leq j \leq n$ and $j \neq i$. Assume $s^{(i,n)} \in C$ for some i with $2 \leq i \leq n-1$. Similarly, $s^{(j)} \notin C$ for $2 \leq j \leq n$ and $j \neq i$. Let c be the center node of C. Then either $c \in G_n(p_n)$ or $c \in G_n(p_i)$. If $c \in G_n(p_n)$, then from $s^{(i,n)} \in C$, $s^{(i,n)} = c^{(n)}$, i.e., $s^{(i)} = c$. From this, C does not block any path $s \to g_{p_j}$ for $j \neq i$. For $c \in G_n(p_i)$, all the nodes of C except $c^{(n)}$ are in $G_n(p_i)$. Since $c^{(n)}$ is a port to

$G_n(p_i)$ and $s^{(j,n)}$ is a port to $G_n(p_n)$, $s^{(j,n)} \notin C$ for $j \neq i$. Therefore, given $\mathbf{F}$ with $|\mathbf{F}| \leq n-1-k$ and $d(\mathbf{F}) \leq 2$, at least k of the $n-1$ paths are fault-free.

Assume r, $r \geq k$, of the $n-1$ paths are fault-free. Obviously, if cluster C blocks a path $s \rightarrow g_{p_i}$, then the center node of C is either in $G_n(p_n)$ or in $G_n(p_i)$. Therefore, at most $r-k$ faulty clusters may have their center nodes in the r substars $G_n(p_j)$ with fault-free path $s \rightarrow g_{p_j}$. Thus, the lemma holds. □

To find k fault-free disjoint paths $t_i \rightarrow h_{l_i} \in G_n(l_i)$, the following preparations are needed.

Lemma 6 *For any node $s = [p_1, p_2, \ldots, p_n]$ in G_n and any $k \in <n> - \{p_1, p_n\}$, there are $n-1$ disjoint paths of length at most 3 that connect s to $n-1$ distinct nodes in $G_n(k)$.*

Proof: Assume that $p_j = k$ for some j with $2 \leq j \leq n-1$. Then the $n-1$ disjoint paths are:

$$P_i : \begin{cases} s \rightarrow s^{(i)} \rightarrow s^{(i,j)} \rightarrow s^{(i,j,n)} \in G_n(k) & \text{if } i \neq j \ (p_i \neq k) \\ s \rightarrow s^{(i)} \rightarrow s^{(i,n)} \in G_n(k) & \text{if } i = j \ (p_i = k) \end{cases}$$

for $2 \leq i \leq n-1$ and

$$P_n : s \rightarrow s^{(n)} \rightarrow s^{(n,j)} \rightarrow s^{(n,j,n)} \in G_n(k).$$

□

Notice that the path P_n in Lemma 6 passes through substar $G_n(p_1)$.

Example 2: Let $s = [2,3,4,1,6,5]$ and $k = 4$. Then the $n-1$ paths are (see Figure 3):

P_2: $s \rightarrow s^{(2)} = [3,2,4,1,6,5] \rightarrow s^{(2,3)} = [4,2,3,1,6,5] \rightarrow s^{(2,3,6)} = [5,2,3,1,6,4]$

P_3: $s \rightarrow s^{(3)} = [4,3,2,1,6,5] \rightarrow s^{(3,6)} = [5,3,2,1,6,4]$

P_4: $s \rightarrow s^{(4)} = [1,3,4,2,6,5] \rightarrow s^{(4,3)} = [4,3,1,2,6,5] \rightarrow s^{(4,3,6)} = [5,3,1,2,6,4]$

P_5: $s \rightarrow s^{(5)} = [6,3,4,1,2,5] \rightarrow s^{(5,3)} = [4,3,6,1,2,5] \rightarrow s^{(5,3,6)} = [5,3,6,1,2,4]$

P_6: $s \rightarrow s^{(6)} = [5,3,4,1,6,2] \rightarrow s^{(6,3)} = [4,3,5,1,6,2] \rightarrow s^{(6,3,6)} = [2,3,5,1,6,4]$

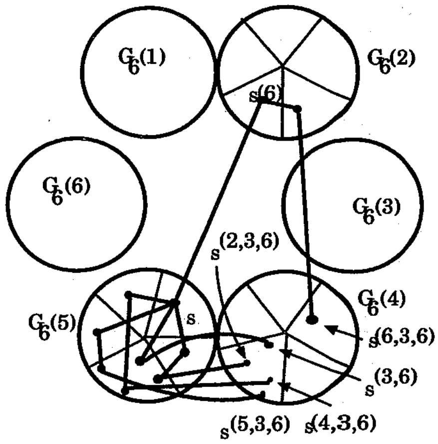

Figure 3: Example 2.

Lemma 7 *Given $s = [p_1, \ldots, p_n]$, $k \in <n> - \{p_1, p_n\}$, and a cluster C with $d(C) \leq 2$, $s \notin C$, and the center node $c \notin G_n(k)$, C can block at most one of the $n-1$ paths P_i, $2 \leq i \leq n$, given in Lemma 6.*

Proof: We first claim that C can not block any two paths P_l and P_m of $P_2, \ldots, P_{n-1}$. Without loss of generality, we assume $P_l : s \rightarrow s^{(l)} \rightarrow s^{(l,j)} \rightarrow s^{(l,j,n)}$ and $P_m : s \rightarrow s^{(m)} \rightarrow s^{(m,j)} \rightarrow s^{(m,j,n)}$ are any two paths of $P_2, \ldots, P_{n-1}$, where $s \in G_n(p_n)$ and $s^{(l,j)}, s^{(m,j)} \in C_n(p_n, k)$. From Lemma 1, $d(s^{(l)}, s^{(m,j)}) \geq 3$. From Lemma 2, $d(s^{(l,j)}, s^{(m,j)}) \geq 3$ and $d(s^{(l,j,n)}, s^{(m,j,n)}) \geq 3$. In general, it is easy to see that for a node $x \in P_l - \{s\}$ and a node $y \in P_m - \{s\}$, if $x = s^{(l)}$ and $y = s^{(m)}$, then $d(x,y) = 2$, otherwise $d(x,y) \geq 3$. On the other hand, for C with $s \notin C$ and $d(C) \leq 2$, from Lemma 1, C can not block both $s^{(l)}$ and $s^{(m)}$. Thus, the claim holds.

Now, we show for C with $d(C) \leq 2$, $s \notin C$, and the center node of C is not in $G_n(k)$, C can not block both P_n and any P_l, $2 \leq l \leq n-1$. Notice that the nodes of $P_n - \{s\}$ are in $G_n(p_1)$ and $G_n(k)$. Assume C blocks some P_l, $2 \leq l \leq n-1$, then the center node c of C is either in $G_n(p_l)$ (in this case $p_l \neq k$) or in $G_n(p_n)$. If $c \in G_n(p_l)$, then $c^{(n)} \in G_n(p_n)$ and path P_n is fault-free. Assume $c \in G_n(p_n)$. If $c^{(n)} \in G_n(j)$ with $j \neq k, p_1$, then obviously P_n is fault-free. If $c^{(n)} \in G_n(p_1)$, then $c^{(n)}$ and $s^{(n)}$ are in the same port-set. From this and Lemma 2, $d(c^{(n)}, s^{(n)}) \geq 3$. Therefore, P_n is fault-free. If $c^{(n)} \in G_n(k)$, then $c^{(n)}$ is port to $G_n(p_n)$ and $s^{(n,j,n)}$ is a port to $G_n(p_1)$, and thus, P_n is fault-free. Similarly, we can prove if C blocks P_n, C does not block any P_l, $2 \leq l \leq n-1$. Thus, the lemma holds. □

Now we give our algorithm for node-to-set CFT routing in Figure 4.

Theorem 8 *For $2 \leq k \leq n-1$, given a set $\mathbf{F}$ of faulty clusters with $|\mathbf{F}| \leq (n-1)-k$ and $d(\mathbf{F}) \leq 2$, and non-faulty nodes $s = [p_1, \ldots, p_n]$ and $T = \{t_1, \ldots, t_k\}$ in G_n, Algorithm Node-To-Set finds in $O(|F| + kn)$ time fault-free disjoint paths $s \rightarrow t_i$ of length at most $d(G_n) + 9$.*

Proof: For $\mathbf{F} = \{C_1, \ldots, C_r\}$, $r \leq n-1-k$, let c_i be the center node of C_i. For each substar $G_n(i)$, $i \in <n>$, let $x_i = |G_n(i) \bigcap \{c_1, \ldots, c_r, c_1^{(n)}, \ldots, c_r^{(n)}\}|$ and $y_i = |G_n(i) \bigcap T|$ (x_i is the number of faulty clusters C with $C \bigcap G_n(i) \neq \emptyset$ and y_i is the number of nodes of T in $G_n(i)$). For each faulty cluster C_i, if the center node c_i appears in a substar $G_n(j)$, then $c^{(n)} \notin G_n(j)$. Therefore, for every $i \in <n>$, $x_i \leq n-1-k$ and $x_i + y_i \leq n-1$. The proof of the theorem is divided into two Cases.

Algorithm Node-To-Set($s, T, \mathbf{F}, G_n$)
Input: Fault clusters $\mathbf{F} = \{C_1, C_2, \ldots, C_r\}$,
$r \leq n-k-1$, non-faulty nodes s and
$T = \{t_1, \ldots, t_k\}$ in G_n.
Output: Fault-free disjoint paths $s \rightarrow t_i$, $1 \leq i \leq k$.
begin
For each C_i, $1 \leq i \leq r$, find the center node c_i of C_i;
For $i \in < n >$, $x_i := |G_n(i) \bigcap \{c_1, \ldots, c_r, c_1^{(n)}, \ldots, c_r^{(n)}\}|$
and $y_i := |G_n(i) \bigcap T|$;
if ($\forall i \in < n >$, $x_i + y_i < n - 1$) **then**
find k substars $G_n(l_i)$ s.t. $G_n(l_i) \bigcap \{c_1, \ldots, c_r\} = \emptyset$,
$l_i \neq p_n$, and $s \rightarrow g_{l_i} \in G_n(l_i)$ is fault-free
else /* $\exists i \in < n >$, $x_i + y_i = n - 1$. */
find k substars $G_n(l_i)$ s.t. $G_n(l_i) \bigcap \{c_1, \ldots, c_r\} = \emptyset$,
$l_i \neq i$, and $s \rightarrow g_{l_i} \in G_n(l_i)$ is fault-free;
Find fault-free disjoint paths $t_i \rightarrow h_{l_i} \in G_n(l_i)$
of length at most 3;
if ($G_n(l_i)$ has at most one node of T) **then**
connect g_{l_i} and h_{l_i} in $G_n(l_i)$ by Lemma 3
else connect g_{l_i} and h_{l_i} in $G_n(l_i)$ by Lemma 4;
end.

Figure 4: Algorithm for Node-to-Set CFT Routing in G_n.

Case 1: $\forall i \in < n >$, $x_i + y_i < n - 1$.
Choose k destination substars $G_n(l_i)$ by Lemma 5 such that $l_i \neq p_n$, $G_n(l_i) \bigcap \{c_1, \ldots, c_r\} = \emptyset$, and there is a fault-free path $s \rightarrow g_{l_i} \in G_n(l_i)$ of length at most 2. If $G_n(l_i)$ has any faulty node, the faulty node is $c^{(n)}$, where c is the center node of some faulty cluster. Therefore, for any fault-free node t with $t^{(n)} \in G_n(l_i)$, $t^{(n)}$ is fault-free. For each $G_n(l_i)$, if $G_n(l_i)$ contains a node t_i, then assign t_i to $G_n(l_i)$ and $h_{l_i} = t_i$ (if $G_n(l_i)$ contains more than one t_i's, then assign any one of t_i to $G_n(l_i)$). Mark $G_n(l_i)$ used. For unused $G_n(l_i)$, if $G_n(l_i)$ contains $t_i^{(n)}$ for some un-assigned t_i, then assign t_i to $G_n(l_i)$. Path $t_i \rightarrow t_i^{(n)} = h_{l_i} \in G_n(l_i)$ is fault-free. Mark $G_n(l_i)$ used. For the rest t_i, assign an arbitrary unused $G_n(l_i)$. Then $t_i, t_i^{(n)} \notin G_n(l_i)$. Assume $t_i \in G_n(m)$. From Lemma 6, there are at least $n-2$ disjoint paths $P_2, \ldots P_{n-1}$ from t_i to $n-2$ distinct nodes in $G_n(l_i)$. We will show that one of the above $n-2$ paths, say P_j, is fault-free. For this P_j, $P_j \bigcap G_n(l) = \emptyset$, $l \in < n > -\{m, l_i\}$, and P_j has a subpath of length at most 2 in $G_n(m)$. If $G_n(m)$ contains other nodes t_j of T, then treating the subpath of $t_j \rightarrow h_{l_j}$ in $G_n(m)$ as a faulty cluster, there are at most $x_m + y_m - 1 < n - 2$ faulty clusters of diameter at most 2 in $G_n(m)$. Therefore, from Lemma 7, a fault-free path $P_j : t_i \rightarrow h_{l_i} \in G_n(l_i)$ of length at most 3 can be found. Obviously, k paths $t_i \rightarrow h_{l_i}$, $1 \leq i \leq k$, found above are disjoint.

For $G_n(l_i)$ which contains at most one node of T, $G_n(l_i)$ may only have some single faulty nodes. Therefore, by Lemma 3, g_{l_i} and h_{l_i} can be connected by a fault-free path of length at most $d(G_{n-1}) + 3$ in $G_n(l_i)$. If $G_n(l_i)$ has at least two nodes of T, then one node t_i is assigned to $G_n(l_i)$ and the other nodes of T are routed out of $G_n(l_i)$. Therefore, $G_n(l_i)$ may have some faulty clusters of diameter 2 due to the routing of other nodes of T. Obviously, $G_n(l_i)$ has at most $x_{l_i} + y_{l_i} - 1 < n - 2$ faulty clusters. By Lemma 4, g_{l_i} and h_{l_i} can be connected by a fault-free path of length at most $d(G_{n-1}) + 8$ in $G_n(l_i)$. The total length of the paths $s \rightarrow t_i$ in this case is at most $\max\{d(G_{n-1})+2+3+3, d(G_{n-1})+2+8\} \leq d(G_n)+9$.
Case 2: $\exists i \in < n >$, $x_i + y_i = n - 1$.
From Lemma 5, we can find at least k destination substars $G_n(l_i)$. If we can choose k $G_n(l_i)$ with $l_i \neq i$, then the destination substars are found. Assume $G_n(i)$ is chosed. From the condition of this case, for any faulty cluster $C \in \mathbf{F}$, $C \bigcap G_n(i) \neq \emptyset$, i.e., $c^{(n)} \in G_n(i)$ for every faulty cluster C. Therefore, we have $G_n(p_n)$ is fault-free (otherwise, $G_n(p_n)$ has at least one center node of faulty clusters, implying that we can choose k $G_n(l_i)$ with $l_i \neq i$). We use $G_n(p_n)$ to replace $G_n(i)$ as a destination substar.

From the condition of this case, for any faulty cluster $C \in \mathbf{F}$, either the center node $c \in G_n(p_n)$ or $c^{(n)} \in G_n(p_n)$. Thus, for any non-faulty node $t \in G_n(i)$, $t^{(n)}$ is fault-free. For each $G_n(l_i)$, if $G_n(l_i)$ contains $t_i^{(n)}$ of some $t_i \in T$, assign t_i to $G_n(l_i)$ (if $G_n(l_i)$ contains more than one $t_i^{(n)}$, assign an arbitrary t_i to $G_n(l_i)$). Mark $G_n(l_i)$ used. For each unassigned node t_i, assign an arbitrary unused $G_n(l_i)$. Since $G_n(l_i)$ does not contain any center node of faulty clusters, from Lemmas 6 and 7, fault-free disjoint paths $t_i \rightarrow h_{l_i} \in G_n(l_i)$ of length at most 3 can be found. For $l_i \neq p_n$, from Lemma 3, g_{l_i} and h_{l_i} can be connected by a fault-free path of length at most $d(G_{n-1}) + 3$, since $G_n(l_i)$ has at most some single faulty nodes. For $l_i = p_n$, we connect h_{l_i} to s. Mark the neighbors $s^{(l)}$, $p_l \neq i$, which is connected to g_{l_j} and then to t_j, as faulty nodes. Then h_{l_i} and s can be connected by a fault-free path of length at most $d(G_{n-1}) + 3$. The path $s \rightarrow h_{l_i}$ is disjoint with the paths $s \rightarrow h_{l_j}$ for $i \neq j$. The length of the paths $s \rightarrow t_i$ in this case is at most $d(G_{n-1}) + 2 + 3 + 3 \leq d(G_n) + 9$.

It takes $O(|F|)$ time to find the centers of faulty clusters. It takes $O(n)$ time to find k destination substars $G_n(l_i)$ by checking the last digits of center nodes of fault clusters. Finding paths $t_i \rightarrow h_{j_i}$ takes $O(kn)$ time. Connecting g_{l_i} and h_{l_i} in $G_n(l_i)$ takes $O(|F_{l_i}| + n)$ time, where $|F_{l_i}|$ is the number of faulty nodes in $G_n(l_i)$. Since $G_n(l_i)$ has originally at most $O(n)$ faulty nodes and routing some nodes t_j out of $G_n(l_i)$ generates at most $O(n)$ new faulty nodes, it takes $O(n)$ time to find one $g_{l_i} \rightarrow h_{l_i}$. Thus, the time complexity of Algorithm Node-To-Set is $O(|F| + kn)$. □

Theorem 8 shows that $(n-1-k, 2)$ is a features number of G_n for node-to-set CFT routing. We now prove $(n-1-k, 2)$ is an OCFT number as well.

Theorem 9 *$(n-1-k, 2)$ is an OCFT number of G_n for node-to-set CFT routing.*

Proof: To show the theorem, we need to prove for any (m,d) with $(n-1-k,2) < (m,d)$, (m,d) is not a features number. Let $s = [p_1, \ldots, p_n]$ and $\mathbf{F} = \{\{[p_i, \ldots, p_{i-1}, p_1, p_{i+1}, \ldots, p_n]\} | 2 \leq i \leq n-k+1\}$. Then $|\mathbf{F}| = n-k$ and $d(\mathbf{F}) = 0 < 2$. Obviously s has only $k-1$ fault-free neighbors. It is impossible to find k fault-free disjoint paths $s \rightarrow t_i$, $1 \leq i \leq k$. Thus, $(n-k,2)$ is not a features number of G_n for node-to-set CFT routing. Similarly, we can show $(n-1-k,3)$ is not a features number as well. □

4 Conclusional Remarks

In this paper, we discussed node-to-set CFT routing in star graphs G_n. For node-to-set CFT routing, we proved $(n-1-k,2)$ is an OCFT number of G_n. We gave $O(|F|+kn)$ time algorithm which find k fault-free routing paths of length at most $d(G_n)+9$ for node-to-set CFT routing. Our result shows that G_n has very good fault tolerant properties in CFT routing. It is interesting to find OCFT number for star graphs for $d(\mathbf{F}) > 2$. Investigating CFT routing properties and designing efficient CFT routing algorithms for other interconnection networks are certainly worth further research attention.

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