Joint Conference of 1996 International Computer Symposium
December 19~21, Kaohsiung, Taiwan, R.O.C.

The Characterizations of Hinge-free Networks

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Abstract

In this paper, a graph representation model for the reliable network designs is presented. A graph is said to be hinge-free if the removal of any single vertex in the graph results in no increase in distance between any two remaining vertices. The advantages of the proposed structure are not only residual connectedness but also distance inheritance. Moreover, a property related to the class of hinge-free graphs and its generalization, called the k-GC graphs (a class of connected graphs by removing at least k vertices is required to increase the distance between any two remaining vertices), is established. Base on this property, we show that all the above classes of graphs can be recognized in $O(|V||E|)$ time in spite of the value of k. We also provide some extensions of graphs with respect to the hinge-free property (resp. k-GC property) such that a hinge-free (resp. k-GC) network on large scale can easily be constructed by using these operations. Finally, the relationships among hinge-free graphs and other classes of graphs are discussed.

1 Introduction

Most of network designs and analysis usually model their topologies as graphical representations in a natural way because that many relevant problems of networks can be solved by using graph theoretic results. As usual, a communication network is modeled as an undirected graph that nodes and edges in the graph correspond to the communication sites and links, respectively. Due to the consideration of immunity in such a system, the designing of reliable graphs has become a major issue. Recently, several variations of invulnerable networks have been proposed but most of them are based on the fundamental feature — all messages transferred among communication sites must be able to complete despite the presence of certain faults [4, 8, 10, 16]. It therefore led to that the reliability measures are concerned mainly with residual connectedness [1, 3, 6, 13]. An important problem about network analysis is how to find the more useful nodes or links whose removal results in a great increase of communication cost to the remaining network. To efficiently design a network with high reliability and low communication cost, in this paper we shall present a class of graphs, called the *hinge-free graphs,* and their corresponding networks that the advantages of the proposed structures are not only residual connectedness but also distance inheritance.

All graphs considered in this paper are assumed to be simple, undirected and connected. Let $G = (V, E)$ be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The cardinalities of $V(G)$ and $E(G)$ are said to be the *order* and the *size*, respectively, of G. Let u and v be two vertices in a graph G. The *distance* between u and v, denoted by $d_G(u, v)$, is the number of edges of a shortest path from u to v in G. A shortest path between u and v is called a *u-v geodesic.* Two paths between u and v are said to be *vertex-disjoint* (resp. *edge-disjoint*) if they do not have any vertex excluding u and v (resp. any edge) in common. The *open neighborhood* (or simply *neighborhood*) $N_G(v)$ of a vertex v is the set $\{u \in V(G) : uv \in E(G)\}$; and the *closed neighborhood* $N_G[v]$ is $N_G(v) \cup \{v\}$. Let $\tau = \langle u, v\rangle$ be a vertex pair of graph G, we say that τ is a *twin pair* if either $N_G(u) = N_G(v)$ or $N_G[u] = N_G[v]$. Moreover, τ is a *true twin* if u and v are adjacent; a *false twin* otherwise. Note that, the subscript G can be dropped in the above definitions when no ambiguity arises. For graph-theoretic terminologies and notations not mentioned here we refer to [15].

In [5], Chang *et al.* defined that a vertex $v \in V(G)$ is a *hinge-vertex* if there exist two vertices $x, y \in V(G - v)$ such that $d_{G-v}(x, y) > d_G(x, y)$, where $G - v$ is the induced subgraph of G with vertex set $V(G) - \{v\}$. It is straightforward to see that the problem of finding all hinge-vertices of an arbitrary graph is polynomial solvable since the time complexity of the well-known all-pairs-shortest-paths problem is polynomial [12, 19]. In fact, a point of view proposed in [5] shows that we only need to inspect the neighbors of vertices instead of examining the distances among all the vertex-pairs to identify the hinge-vertices for an arbitrary graph. We include this property in Lemma 2.1. Base on this property, the linear time algorithms for finding all hinge-vertices of strongly chordal graphs and permutation graphs can be found in [5, 17].

In this paper, we shall introduce a specific class of graphs characterized by the absence of hinge-vertex. A graph without hinge-vertex is called a *hinge-free graph* and its corresponding network is called a *hinge-free network*. Another synonym of hinge-free graph is called the *self-repairing graph* by Farley and Proskurowski [11]. In this paper, we use the more descriptive term — hinge-free. Obviously, every hinge-free graph is a biconnected graph. By definition, we know that a hinge-free graph permits one node fault without increasing the distance among all the remaining nodes. Generally, the definition of hinge-free graphs may be extended to a further permission with k faulty nodes for $k \geq 2$. In [9], Entringer *et al.* defined a graph G to be *k-geodetically connected* (k-GC for short) if G is connected and the removal of at least k vertices is required to increase the distance between any pair of remaining vertices. That is, G is a k-GC graph if and only if $d_{G-S}(x,y) = d_G(x,y)$ for every pair of vertices $x,y \in V(G) - S$, where S is any subset of $V(G)$ with $|S| \leq k-1$ vertices. Clearly, the class of hinge-free graphs is equivalent to the class of 2-GC graphs, and all the classes of GC graphs constitute a hierarchy by set inclusion, i.e., $(k+1)$-GC $\subset$ k-GC. Besides, another two interesting subclasses of k-GC graphs will also be mentioned in this paper. A k-GC graph G is *minimal* if the resulting graph of removing any edge in G is not a k-GC, and G is *minimum* if G has the minimum size when its order is given. Obviously, every minimum k-GC graph must be minimal, but the converse is not true. For the special case $k = 2$, we use the terms *minimal hinge-free* and *minimum hinge-free*.

Similarly, the notion of hinge-free may be concerned in the edges for a graph. Let G be a graph. An edge $e \in E(G)$ is said to be a *hinge-edge* if there exist two nonadjacent vertices $x,y \in V(G)$ such that $d_{G-e}(x,y) > d_G(x,y)$, where $G-e$ denotes the graph by removing e from G. That is, every x-y geodesic in G must pass through e. A graph is *edge hinge-free* if it contains no hinge-edge. Generally, we define a graph G to be *k-geodetically edge-connected* (k-GEC for short) if G is connected and for any pair of nonadjacent vertices u and v in G, at least k edges must be deleted to cut all u-v geodesics. Obviously, the structure of k-GEC graphs can tolerate any $k-1$ faulty links without increasing the distance of any two nonadjacent vertices.

In [9], Entringer *et al.* showed that G is a k-GC graph if and only if every two vertices in G with distance 2 apart are joined by at least k vertex-disjoint geodesics. It is interesting to know that if there is a similar property for the k-GEC graphs. In this paper we give another proof to acquire Entringer's property and show that the answer is affirmative. Moreover, the proposed result exhibits that the above two classes of graphs are equivalent and there are at least k common neighbors between any pair of vertices with distance 2. Base on this characterization, we will show that all the k-GC graphs for arbitrary k can be recognized in $O(|V||E|)$ time.

Let G be a graph associated with a certain property Π. An *extension* of G with respect to Π is an operation for constructing a supergraph G' from G by adding at least one vertex such that G' satisfies Π. The study of graph extensions can be applied to the real life applications of distributed network designs because that it allows the configuration of networks to expand in scale without destroying a given property. In [11], several kinds of extensions with respect to the hinge-free property have been defined. In this paper, we will generalize their results to the k-GC property for arbitrary k. In addition, we shall introduce another extension, called the *cross extension*, for combining two k-GC graphs to from a new one and preserving the k-GC property.

The relationships among various classes of graphs had been individually investigated by many researches and can be referred to [7, 14, 20]. This, quite naturally, has motivated the study of relationships between hinge-free graphs and other classes of graphs. In [18], Howorka defines a graph G to be *distance-hereditary* if every pair of vertices in any connected induced subgraph of G inherits its distance from G. Intuitively, there is a similarity between the definitions of hinge-free graphs and distance-hereditary graphs. The discussion of relationships between (minimum) hinge-free graphs and several classes of graphs including distance-hereditary, biconnected, bipartite, and (6,2)-chordal bipartite will be covered in this paper.

2 The recognition algorithm

Lemma 2.1 was provided in [5] to identify the hinge-vertex of a general graph. A similar property to identify the hinge-edge of a general graph is shown in Lemma 2.2.

Lemma 2.1 *For a graph $G = (V,E)$, a vertex $v \in V(G)$ is a hinge-vertex of G if and only if there exist two nonadjacent vertices $x,y \in N(v)$ such that $N(x) \cap N(y) = \{v\}$.*

Lemma 2.2 *Let $G = (V,E)$ be a graph and $uv \in E(G)$. uv is a hinge-edge of G if and only if at least one of the following conditions hold:*

(1) *There is a vertex $w \in N(u)$ and $w \notin N(v)$ such that $N(w) \cap N(v) = \{u\}$.*
(2) *There is a vertex $w \in N(v)$ and $w \notin N(u)$ such that $N(w) \cap N(u) = \{v\}$.*

Proof. The sufficient condition is obvious from the definition. We prove the necessary condition as follows. Suppose that $e = uv$ is a hinge-edge of G. By definition, there exist two nonadjacent vertices, say x and y, in $V(G)$ such that every x-y geodesic must pass through e. Let $P = x \cdots uv \cdots y$ be a geodesic between x and y. Since x and y are nonadjacent, the two end vertices of e cannot be x and y. This indicates that there is a vertex w on P adjacent to u or v but not both. For the case $wu \in E(G)$ and $wv \notin E(G)$, we can see that u must be the only vertex adjacent to both w and v; otherwise, there is another x-y geodesic which does not pass through e. For the case $wv \in E(G)$ and $wu \notin E(G)$, $N(w) \cap N(u) = \{v\}$ can be proved by a similar reasoning. □

Theorem 2.3 *Let G be a graph and $k \geq 2$ be an integer. The following statements are equivalent:*

(1) *G is a k-GC graph.*

(2) *Every pair of nonadjacent vertices in G are joined by at least k vertex-disjoint geodesics.*

(3) *Every pair of nonadjacent vertices in G are joined by at least k edge-disjoint geodesics.*

(4) *G is a k-GEC graph.*

(5) *Every pair of vertices $u, v \in V(G)$ with $d(u,v) = 2$ satisfies $|N(u) \cap N(v)| \geq k$.*

Proof. The implications (2)⇒(1) and (2)⇒(3)⇒(4) are trivial. To complete the proof we only need to show (1)⇒(4)⇒(5)⇒(2) as follows.

(1)⇒(4) Assume that G is not a k-GEC graph. By definition, there exist two nonadjacent vertices $u, v \in V(G)$ and a set $F \subset E(G)$ with $|F| < k$ such that $d_{G-F}(u,v) > d_G(u,v)$. Let $S \subset V(G)$ be a minimum set of vertices which covers all the edges of F. Clearly, $|S| \leq |F| < k$ and $d_{G-S}(u,v) \geq d_{G-F}(u,v) > d_G(u,v)$. Thus G is not a k-GC graph.

(4)⇒(5) Let G be a k-GEC graph and assume that there exist two vertices $u, v \in V(G)$ with $d_G(u,v) = 2$ satisfying $|N_G(u) \cap N_G(v)| < k$. Let $S = N_G(u) \cap N_G(v)$ and $F = \{uw \in E(G) : w \in S\}$. Clearly, $|F| = |S| < k$ and $d_{G-F}(u,v) > d_G(u,v) = 2$. This contradicts that G can tolerate any $k-1$ faulty edges without increasing the distance of any two nonadjacent vertices.

(5)⇒(2) The proof is produced by induction on the distance d between two nonadjacent vertices of G. The case with $d = 2$ is trivial. Assume that every pair of nonadjacent vertices with distance $d \leq l$ in G are joined by at least k vertex-disjoint geodesics. Then, we consider two nonadjacent vertices u and v in G with distance $d(u,v) = l+1$ and show that there are at least k vertex-disjoint geodesics between u and v in G. Let P be a u-v geodesic and x be a vertex on P adjacent to v. By the assumption, u and x are joined by at least k vertex-disjoint geodesics with length l. So that we can find a set of vertices $\{w_i : 1 \leq i \leq k\} \subset N(x)$ such that each w_i is in a u-x geodesic. Also we can see that each w_i and v must be nonadjacent; otherwise, $d(u,v) < l+1$. Thus, $d(w_i, v) = 2$ and $|N(w_i) \cap N(v)| \geq k$ for $i = 1, \ldots, k$. This indicates that we have w_i-v vertex-disjoint geodesics in G for all i, and therefore u and v must be joined by at least k vertex-disjoint geodesics with length $l+1$. □

Theorem 2.3 suggests that a k-GC graph recognition algorithm can easily be implemented as follows. For an arbitrary input graph G and a given integer $k \geq 2$, we explore the vertices of G from each vertex $v \in V(G)$ as the root by using breadth-first search. When a breadth-first search arrives at some vertex u with $d(v,u) = 2$, an auxiliary counter of u is used to record the times of u visited. An exploration is terminated if all the vertices of distance 2 from v have been visited. Thus, if there is a counter of some vertex u whose value is less than k then G cannot be a k-GC graph. After all the explorations completed, we can determine if G is a k-GC graph. Besides, for each iteration rooted at v, the counter of u is recorded by 1 if and only if the common neighbor of u and v is a hinge-vertex of G. Assume that w is the common neighbor of u and v in this case. By Lemma 2.2, uw and vw are two hinge-edges of G. Since every vertex of G is chosen as root for once, the approach can find all the hinge-vertices and hinge-edges of G in $O(|V||E|)$ time. Hence, we have following theorem.

Theorem 2.4 *Given a graph G and an integer $k \geq 2$, there is an algorithm to recognize if G is a k-GC graph in $O(|V||E|)$ time. Moreover, if G is not a 2-GC graph then the algorithm can identify all the hinge-vertices and hinge-edges of G in the same time complexity.*

3 Some Extensions of k-geodetically connected Graphs

In [11], the authors show that the class of hinge-free graphs has an excellent property on extensions. In this section, we will generalize their results to the classes of k-GC graphs for arbitrary k.

3.1 The isosceles extension

Let $\langle u, v \rangle$ be a twin pair of a graph G. The following properties must be satisfied: (1) for any other vertex $w \in V(G)$, $d(u,w) = d(v,w)$; (2) $d(u,v) \leq 2$; and (3) both u and v are not hinge-vertices. Furthermore, if $\langle u, v \rangle$ is a true twin of G then any geodesic between two nonadjacent vertices in G cannot pass through edge uv. So that every minimal hinge-free graph with order ≥ 4 contains no true twin. An *isosceles extension* with respect to the twin pair $\langle u, v \rangle$ is a one vertex extension that a new graph G' is obtained from G by adding an attaching vertex w to G and making w adjacent to u and v in G'. For example, Figure 1 (a) shows that a graph G has a false twin $\langle v_2, v_3\rangle$ and a true twin $\langle v_4, v_5\rangle$. Two isosceles extensions with respect to $\langle v_2, v_3\rangle$ and $\langle v_4, v_5\rangle$ in G are shown in Figure 1 (b) and 1 (c), respectively.

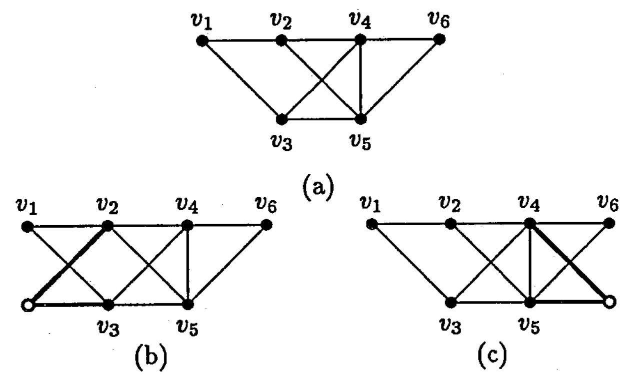

Figure 1: (a) an example to illustrate the twin pairs of graph G (b),(c) the isosceles extensions of G.

Base on the result of Theorem 2.3, it is easy to see that if G is a hinge-free graph then any isosceles extension of G must be hinge-free. In fact, Farley and Proskurowski [11] have mentioned an interesting result acquired from their recent works. They show that all the minimum hinge-free graphs with order > 4 excluding Q_3, a 3-regular cube graph on 8 vertices, can be generated from C_4 by a sequence of isosceles extensions. In their paper, the class of these graphs together with C_4 are called *twin graphs*. Hence, we have the following observations: (1) Every minimum hinge-free graph with order ≥ 4 is bipartite. (2) A hinge-free graph G with order $p \geq 4$ is minimum if and only if the size of G is precisely $2p-4$.

Now we expand the isosceles extension from two edges increment to multiple edges increment to construct the k-GC graphs on large scale. To simplify the description we need following definitions: let G be a graph and S be a subset of $V(G)$ with $k = |S| \geq 2$ vertices. S is called a ***k*-size multiplet** of G if for any two vertices $u, v \in S$ in G, either $N(u) = N(v)$ or $N[u] = N[v]$. For a special case with $k = 2$, S is called a twin pair in the previous definition. A ***k*-isosceles extension** of a graph G is a one vertex extension that a new graph G' is obtained from G by adding an attaching vertex w to G such that $V(G') = V(G) \cup \{w\}$ and $E(G') = E(G) \cup \{wu : u \in S\}$, where S is a k-size multiplet of G. In the above construction, we can see that any pair of vertices with distance 2 in G' has at least k common neighbors. Bases on the result of Theorem 2.3, we have a natural extension as follows:

Theorem 3.1 *The class of k-GC graphs is closed under k-isosceles extension.*

For example, the complete bipartite graph $K_{n,n}$ is an n-GC graph for $n \geq 2$ since every pair of nonadjacent vertices in $K_{n,n}$ are joined by n vertex-disjoint geodesics. Clearly, every vertex in an n-GC graph must have at least n neighbors. Thus the size of an n-GC graph with order $2n$ cannot less than n^2. It is easy to see that $K_{n,n}$ for $n \geq 2$ is a minimum n-GC graph. Now we consider a graph G with order $p > 2n$ that is obtained from $K_{n,n}$ by a sequence of n-isosceles extensions for $n \geq 2$. By Theorem 3.1, we know that G is an n-GC graph. Moreover, the size of G is precisely $np - n^2$. It is easy to see that the class of twin graphs is just a special case for $n = 2$.

3.2 The cloning extension and double extension

A *cloning extension* is a one vertex extension that the attaching vertex has the same neighbors with some vertex of original graph. A *double extension* is a composition of two copies of original graph by connecting a bridging edge between each corresponding pair of vertices in these two copies. Figure 2 is helpful in visualizing these two operations. The following two theorems were provided in [11] for constructing a larger hinge-free network, and a routing scheme based on these two operations can be determined.

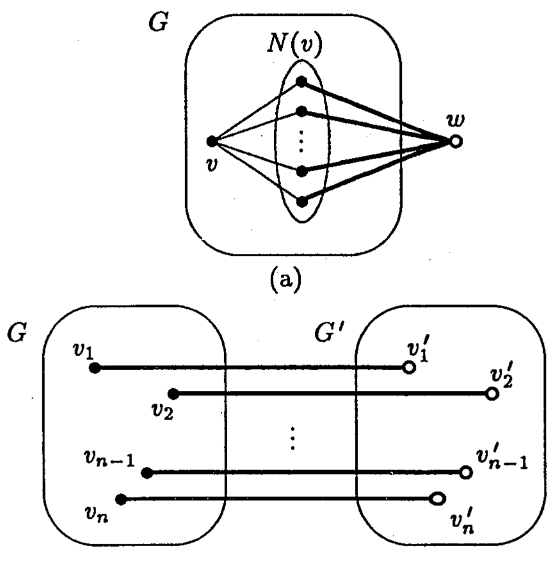

Figure 2: (a) cloning extension (b) double extension.

Theorem 3.2 *If G is a hinge-free graph then any cloning extension of G is hinge-free.*

Theorem 3.3 *If G is a hinge-free graph (resp. minimal hinge-free graph) then the double extension of G is also hinge-free (resp. minimal hinge-free).*

For example, $K_{2,3}$ is a hinge-free graph. $K_{3,3}$ is also hinge-free since it can be obtained from $K_{2,3}$ by a cloning extension. In general, all the complete bipartite graph $K_{n,m}$ for $n, m \geq 2$ are hinge-free. Also we can see that the cloning extension does not maintain the minimal hinge-free property. Clearly, $K_{2,3}$ is minimal hinge-free but $K_{3,3}$ is not. In fact, the number of additional edges of a cloning extension is not a constant. Besides, Q_3 is a minimal hinge-free graph which can be constructed from C_4 by a double extension. Furthermore, iteratively carry out such an operation, the class of graphs known as hypercubes can be generated. Thus, each Q_n for $n \geq 3$ is minimal hinge-free which follows the result of Theorem 3.3. Note that a minimum hinge-free graph with order $p \geq 4$ has size $2p-4$. For the case Q_n, $n \geq 4$, is not minimum hinge-free which can easily be verified. So that the double extension does not maintain the minimum hinge-free property.

Given two graphs G_1 and G_2, the *Cartesian product* $G_1 \times G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$ and any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Obviously, for any graph G the double extension is just the product graph $G \times K_2$ and K_2 is trivially hinge-free. In what follows, we will show that the result of Theorem 3.3 can be generalized.

Theorem 3.4 *If G_1 and G_2 are two hinge-free graphs (resp. minimal hinge-free graphs) then $G_1 \times G_2$ is also hinge-free (resp. minimal hinge-free).*

Proof. First we show that $G_1 \times G_2$ is hinge-free. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two vertices of $G_1 \times G_2$. By the definition of Cartesian product, clearly, $d_{G_1 \times G_2}(u, v) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$. We then consider following two cases:

Case 1: $u_1 = v_1$ or $u_2 = v_2$. In this case, both u and v must be in one copy of G_1 or G_2. Since both G_1 and G_2 are hinge-free and every geodesic connects u and v in G_1 or G_2 is still a u-v geodesic in $G_1 \times G_2$, we guarantee that there are at least two vertex-disjoint u-v geodesics in $G_1 \times G_2$. By Theorem 2.3, $G_1 \times G_2$ is hinge-free.

Case 2: $u_1 \neq v_1$ and $u_2 \neq v_2$. Consider the two vertices $x = (u_1, v_2)$ and $y = (v_1, u_2)$ of $G_1 \times G_2$. In this case, we choose u-x and y-v geodesics from distinct copies of G_2 and x-v and u-y geodesics from distinct copies of G_1, respectively. Then, we have two vertex-disjoint paths between u and v in $G_1 \times G_2$. In addition, the length of path between u and v passing through x (resp. y) in $G_1 \times G_2$ is $d_{G_1 \times G_2}(u, x) + d_{G_1 \times G_2}(x, v) = d_{G_2}(u_2, v_2) + d_{G_1}(u_1, v_1) = d_{G_1 \times G_2}(u, v)$ (resp. $d_{G_1 \times G_2}(u, y) + d_{G_1 \times G_2}(y, v) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2) = d_{G_1 \times G_2}(u, v)$). Thus we find that there are at least two vertex-disjoint u-v geodesics in $G_1 \times G_2$. So that the result follows from Theorem 2.3.

Let G_1 and G_2 be two minimal hinge-free graphs. To show that the minimal hinge-free property is preserved in $G_1 \times G_2$, we need to show that $G_1 \times G_2 - e$ is not hinge-free for any $e \in E(G_1 \times G_2)$. Assume that u and v are any two adjacent vertices of $G_1 \times G_2$. Then, we consider the removal of uv from $G_1 \times G_2$ by following two cases:

Case $1'$: $u_1 = v_1$ and $u_2v_2 \in E(G_2)$. In this case, uv must be contained in one copy of G_2. Let H be such a copy of G_2 and $H' = H - uv$. Since H is minimal hinge-free, the removal of uv means that H' is not hinge-free. Let w be a hinge-vertex of H'. By Lemma 2.1, there exist two nonadjacent vertices $x, y \in N_{H'}(w)$ such that xwy forms a unique x-y geodesic in H'. Also, it is easy to see that the path joined x and y passing through another copy of G_2 adds the length at least 2 to the distance $d_{H'}(x, y)$. This indicates that another x-y geodesic in $G_1 \times G_2 - uv$ cannot exist. Thus $G_1 \times G_2 - uv$ is not a hinge-free graph. Therefore, the edge uv within $G_1 \times G_2$ must be required to maintain the minimal hinge-free property.

Case $2'$: $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. The proof is similar to the case $1'$. □

Now we consider the cloning extension on a k-GC graph G. Let G' be the new graph obtained from G by a cloning extension and let w be the attaching vertex. Because that w has the same neighbors with some vertex u of G, and every pair of vertices $u, v \in V(G)$ with $d_G(u, v) = 2$ satisfies $|N_G(u) \cap N_G(v)| \geq k$, this implies that $|N_{G'}(w) \cap N_{G'}(v)| \geq k$ for any vertex $v \in V(G')$ with $d_{G'}(w, v) = 2$. Thus, we can easily extend the result of Theorem 3.2 to the following theorem:

Theorem 3.5 *The class of k-GC graphs is closed under cloning extension.*

To deal with the double extension and its generalization, the Cartesian product, on a k-GC graph for $k > 2$, a surprising result shows that the closure property cannot preserve. For example, $K_{3,3}$ is a 3-GC graph and not all the pairs of nonadjacent vertices in the graph $K_{3,3} \times K_2$ are joined by at least 3 vertex-disjoint geodesics.

3.3 The cross extension

As we have seen that the cloning extension and double extension cannot guarantee the minimum hinge-free property preserved. Moreover, the use of double extensions caused the number of vertex increment to be a multiple of original graph. In general, supporting irregular extensibility of topology designs is a very essential and desirable property for distributed networks. Hence, we shall introduce a more adequate operation, called the *cross extension*, for combining two minimum hinge-free graphs to form a large one. Generally, we may define this operation on two k-GC graphs for $k \geq 2$ and call the *k-cross extension*. Let G_1 and G_2 be two k-GC graphs for $k \geq 2$. Note that they may be not identical. Assume that S_1 and S_2 are any two k-size multiplets, respectively, of G_1 and G_2. The k-cross extension of G_1 and G_2 connects these two graphs to form a new graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in S_1, v \in S_2\}$. Figure 3 illustrates the operation for $k = 2$. Thus, the topological designs for network extensions using cross extension is more flexible than using the double extension.

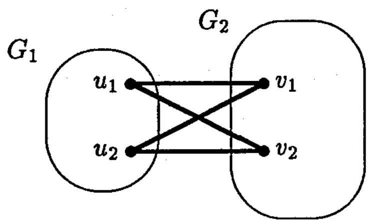

Figure 3: The cross extension.

In the above construction, it is easy to see that there are at least k common neighbors between any pair of vertices with distance 2 in G. So that, the closure property of k-cross extension follows the result of Theorem 2.3. Since every minimum hinge-free graph with order $p \geq 4$ has size $2p-4$, if we restrict G_1 and G_2 to be two minimum hinge-free graphs that each has order ≥ 4, the result of Theorem 3.7 is immediately derived.

Theorem 3.6 *The class of k-GC graphs is closed under k-cross extension.*

Theorem 3.7 *If G_1 and G_2 are two minimum hinge-free graphs each has order ≥ 4, then the cross extension of G_1 and G_2 is minimum hinge-free.*

4. Relationships Between Hinge-free Graphs and Other Graphs

In this section, the relationships among several classes of graphs including hinge-free, minimum hinge-free (MHF), distance-hereditary, bipartite and (6,2)-chordal bipartite are considered. We illustrate the relationships in Figure 4. In this figure, the corresponding region of each specific class of graphs is indicated by a sequence of regional numbers which are surrounded with parentheses. For more detailed information, see Table 1.

A graph G is called *distance-hereditary* if each connected induced subgraph H of G has the property that $d_H(u,v) = d_G(u,v)$ for each $u, v \in V(H)$. The class of these graphs was first introduced by Howorka [18] and further characterized by Bandelt and Mulder [2]. A bipartite graph G is *(k,l)-chordal* if each cycle of length $\geq k$ in G contains at least l chords. The (6,2)-chordal bipartite graphs are precisely bipartite distance-hereditary graphs that has been independently proved in [2, 7]. Two interesting characterizations related to the structures of these two classes of graphs are stated as follows:

Theorem 4.1 *graph G is distance-hereditary if and only if it can be generated from K_2 by a sequence of the following extensions:*

(1) *type α – adding a new vertex v' and joining it to one vertex v in G.*
(2) *type β – adding a new vertex v' and joining it to all the vertices of $N[v]$ for some vertex v in G.*
(3) *type γ – adding a new vertex v' and joining it to all the vertices of $N(v)$ for some vertex v in G.*

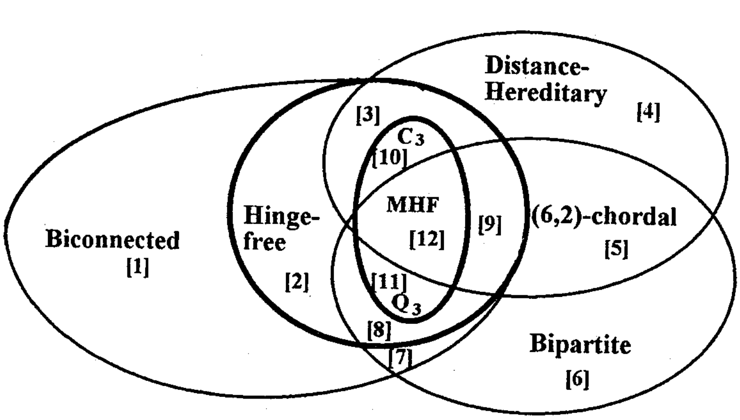

Figure 4: Relationships among hinge-free graphs, minimum hinge-free graphs (MHF) and other graphs.

Table 1: A summary table to illustrate the relationships of Figure 4.

regional number	biconnected	hinge-free	distance hereditary	bipartite	(6,2)-chordal bipartite	minimum hinge-free	twin graph	example
1	√							C_5
2	√	√						$K_3 \times K_2$
3	√	√	√					$K_4 - e$
4			√					Fig. 5(a)
5			√	√	√			$K_{1,3}$
6				√				Fig. 5(b)
7	√			√				C_6
8	√	√		√				Fig. 5(c)
9	√	√	√	√	√			$K_{3,3}$
10	√	√	√			√		C_3
11	√	√		√		√		Q_3
12	√	√	√	√	√	√	√	C_4

A mark √ indicates that a specific region is associated with the property of a certain class of graphs.

Theorem 4.2 *A graph G is $(6,2)$-chordal bipartite if and only if it can be generated from K_2 by a sequence of types α and γ extensions.*

Note that the γ extension is just the cloning extension defined in the previous section. The definitions of hinge-free graphs and distance-hereditary graphs seem to be similar by intuition, whereas they are not properly contained to each other. However, not all distance-hereditary graphs are biconnected. A biconnected graph that is hinge-free but not distance-hereditary can be seen in Figure 5 by considering $G = K_3 \times K_2$ and the induced subgraph $H = G - \{v_2, v_4\}$, where $d_G(v_1, v_3) = 2 < d_H(v_1, v_3) = 3$. Let v be any vertex of a biconnected distance-hereditary graph G. Clearly, $G - v$ is connected and $d_{G-v}(x, y) = d_G(x, y)$ for any two vertices $x, y \in V(G - v)$. Thus we have following observation:

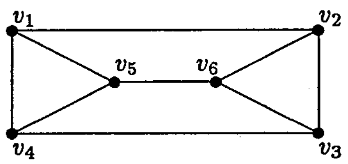

Figure 5: An example of hinge-free graph but not distance-hereditary.

Observation 4.3 *All the biconnected distance-hereditary graphs are hinge-free.*

Now we consider the relationships among minimum hinge-free graphs and other classes of graphs. Clearly, C_4 is a (6,2)-chordal bipartite graph. The following theorem and corollary show that the characterizations of distance-hereditary and (6,2)-chordal bipartite are preserved under isosceles extensions. Recall that the class of minimum hinge-free graphs with order $p \geq 3$ excluding C_3 and Q_3 is equivalent to the class of twin graphs, and all the twin graphs can be generated from C_4 by a sequence of isosceles extensions [11]. This implies that the class of twin graphs must be contained in the class of (6,2)-chordal bipartite. In addition, it is easy to verify that C_3 is distance-hereditary but not bipartite, and Q_3 is bipartite but not distance-hereditary. Therefore, the regions with number 10 and 11 shown in Figure 4 contain exactly C_3 and Q_3, respectively.

Theorem 4.4 *The class of distance-hereditary graphs is closed under isosceles extensions.*

Proof. Let $\tau = \langle u, v \rangle$ be a twin pair of distance-hereditary graph G. By definition, the induced subgraph $G - v$ remains distance-hereditary. Now we consider that a $\langle u, v \rangle$ isosceles extension adds an attaching vertex w into G to form a new graph G'. Then, G' may also be generated from $G - v$ by following two consecutive operations:

(1) join w and u by α extension, and

(2) join v to the vertices of $N[u]$ by β extension if τ is a true twin, or join v to the vertices of $N(u)$ by γ extension otherwise.

Therefore, G' is distance-hereditary. □

In the above proof, if we restrict τ to be a false twin and take (6,2)-chordal bipartite graph in place of distance-hereditary graph, then following corollary can be achieved.

Corollary 4.5 *The class of $(6,2)$-chordal bipartite graphs is closed under isosceles extensions.*

To show that each region in the pictorial relationships of Figure 4 is not empty, we provide some example of graphs that each is contained in a bounded region. Figure 6 (a) shows a graph that is distance-hereditary but not biconnected and bipartite. Figure 6 (b) shows a graph that is bipartite but not distance-hereditary and biconnected. Figure 6 (c) shows a graph that is bipartite hinge-free but not (6,2)-chordal and minimum hinge-free. We make a summary of the above presentation in Table 1.

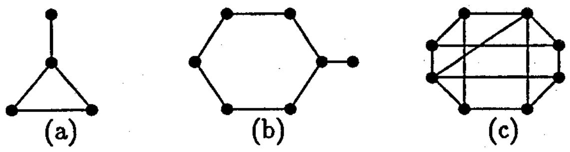

Figure 6: Some illustrating example of graphs in Table 1.

5. Conclusions

In this paper, a hinge-free topology applied to the communication network designs is concerned. Finally, two open problems are given as follows:

(1) The *optimally hinge-free graphs completion problem.* For an arbitrary graph G, the optimally hinge-free graphs completion problem is to find a minimum cardinality set $E' \subseteq E(\bar{G})$, where $\bar{G}$ is the complement of G, such that a new graph with vertex set $V(G)$ and edge set $E(G) \cup E'$ results in a hinge-free graph. In our recent works, we have known some kinds of constructions to design the optimally hinge-free graphs such as paths, cycles and trees, but if it is in a more complicated structure then a comprehensive approach of solving this problem is still open.

(2) The *minimum hinge-free graphs enumeration problem.* An interesting problem related to the combinatorial enumeration is how many minimum hinge-free graphs exist when the number of vertices is given. Let N_p denote the number of p-vertex minimum hinge-free graphs. Because that every minimum hinge-free graph with order $p > 4$ excluding Q_3 can be generated from another minimum hinge-free graph with order $p-1$ and contains $2p-4$ edges, this suggests that maybe N_p can be derived from a recurrence formula only with respect to p. Assume that this is true, we conjecture that the ratio $N_p \,/\, N_{p+1}$ for p even or odd is converged into a fixed value when p tends to infinity.

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