FAULT DIAMETER OF THE CAYLEY GRAPHS BASED ON THE ALTERNATING GROUP

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Abstract

The paper computes the exact fault diameter of the Cayley graphs based on the alternating group. The fault diameter of a graph is the maximum diameter achieved when we delete from the original graph any set of nodes that is smaller than its node connectivity. Based on the algebraic properties of the generators, we show that the fault diameter is the original diameter plus one.

Keywords: Interconnection networks, Cayley graphs, alternating group, fault diameter

1. INTRODUCTION

Group graphs provide a rich framework for the design of the topology of interconnection networks. Famous network topologies such as hypercube[4], star graphs[1], etc. are all able to be modeled as Cayley graphs[2]. Alternating group graphs[8], introduced by Jwo, Lakshmivarahan and Dhall, are Cayley graphs based on the alternating groups. They have shown that alternating group graphs are edge symmetry, 2-transitive, Hamiltonian and strongly hierarchical. They also describe algorithms for embedding of a class of multidimensional grids with unit expansion and dilation three and embedding of a variety of cycles with unit dilation. The simulations of greedy routing on hypercubes, star graphs and alternating group graph have been evaluated. It shows that alternating group graphs offer the best performance with respect to delay time and maximum queue length.

The alternating group graphs are defined as follows. Let $<n>=\{1, 2, \ldots, n\}$, $p = p_1p_2\ldots p_n$, $p_i \in <n>$ and $p_i \neq p_j$ for $i \neq j$, where p_i denotes the element at position i for $1 \leq i \leq n$. That is, p is a permutation of $<n>$. The permutation p can also be represented by its cycle structure as

$$p = c_1c_2\ldots c_ke_1e_2\ldots e_l,$$

where c_i is a *cycle* of length $|c_i| \geq 2$ for $1 \leq i \leq k$ and e_i is an invariant for $1 \leq i \leq l$. Thus, $n = \sum_{i=1}^{k} |c_i| + l$. For convenience, the set of all invariants may be omitted in the cycle representation of p. For example, the cycle structure of permutation 132546 is (23)(45)(1)(6), where (1) and (6) are invariants and may be omitted. Let $\zeta(p)$ denote the number of inversions in p. The parity of p is defined as $\eta(p) = (-1)^{\zeta(p)}$. A permutation is called *even* or *odd* depending on its parity being +1 or –1. Thus, 132546 is an even permutation.

Let S_n be the symmetric group, i.e., S_n contains all the permutations of n elements. The alternating group A_n contains the set of all even permutations of S_n, where $|A_n| = n!/2$. Let $g_i+ = (1\ 2\ i)$, $g_i- = (1\ i\ 2)$ and $\Omega = \{g_i+ \mid 3 \leq i \leq n\} \cup \{g_i- \mid 3 \leq i \leq n\}$. It is well known that Ω is a generator set for A_n.

Definition 1. An alternating group graph of dimension n, $AG_n = (V_n, E_n)$, is defined as

$V_n = A_n$, the set of all even permutations of $<n>$

and

$E_n = \{(p, q) \mid p, q \in A_n, q = p \cdot h, \text{ for } h \in \Omega\}$,

where "$\cdot$" is the usual binary combination operator defined as $f \cdot g(x) = f(g(x))$.

An important parameter for graphs is their *fault tolerance* and *fault diameter*. The fault tolerance of a graph is defined as the maximum number of nodes that can be removed from it provided that the remaining graph is still connected. Hence, the fault tolerance of a graph is defined to be one less than its connectivity. By the theorems proved in [5], the connectivity of the alternating group graphs is optimal, i.e., equal to the degree.

Fault diameter d^f of the graph G with fault tolerance f is defined as the maximum diameter of any graph obtained from G by deleting at most f nodes. If an interconnection network is to be used as a multicomputer communication medium, it is very important that its fault diameter is close to its normal diameter. A family of graphs $\{G_n\}$ is defined to be *strongly resilient* if the fault diameter d_n^f of any member of the family G_n is at most $d_n + k$, where d_n is the normal diameter of G_n and k is a constant independent of n.

We will show that for the alternating group graphs, the fault diameter is the original diameter plus one.

2. BACKGROUND

The n dimensional alternating group graph AG_n is a regular graph with degree $2(n-2)$, $|V_n| = n!/2$, $|E_n| = (n-2) \times n!/2$, diameter $d_n = \lfloor 3 \times (n-2)/2 \rfloor$, and the connectivity is $2(n-2)$. Alternating group graphs have a highly recursive structure. AG_n is made up of n copies of AG_{n-1}.

An understanding of the routing algorithm is essential for our development. Let s, $d \in A_n$. By definition, $(s, d) \in E_n$ if and only if $d = s\cdot h$, $h \in \Omega$. By extending the notion of an edge in AG_n, the path from s to d can be represented by a sequence of generators:

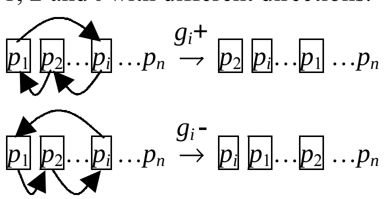
$$\text{Path}(s, d) = h_1 \cdot h_2 \cdots h_t,$$

that is, $d = s \cdot h_1 \cdot h_2 \cdots h_t$, where $h_i \in \Omega$ for $1 \leq i \leq t$. By group theory, we have

$$I = d^{-1} \cdot s \cdot h_1 \cdot h_2 \cdots h_t.$$

Let $p = d^{-1} \cdot s$. Then the properties of path from s to d can be analyzed by considering the path from p to I. So we can focus how to route from p to I only, and the routing is equivalent to sorting a permutation.

Consider what is changed by a routing step on g_i+ and g_i-, $3 \leq i \leq n$. Clear, they rotate the three symbols at position 1, 2 and i with different directions.



As shown in the above figure, g_i+ moves the symbol at position i to position 2, the symbol at position 2 to position 1, and the symbol at position 1 to position i; while g_i- moves the symbol at position i to position 1, the symbol at position 1 to position 2, and the symbol at position 2 to position i. Obviously, $g_i{+}g_i{+} = g_i{-}$, $g_i{-}g_i{-} = g_i{+}$ and $(g_i+)^{-1} = g_i-$, $3 \leq i \leq n$. To find the shortest path from p to I is to optimally *sort* the set of symbols in p using the basic rotations. (We say symbol i is sorted if it is at position i.) Notice that there is no need to sort symbol 1 and 2, they will be automatically sorted after sorting symbol 3, 4, …, n because it should be an even permutation.

The shortest path routing algorithm from $p = p_1p_2\ldots p_n$ to I may be described as follows: Let p' be the first node on the shortest path from p to I. Then,

(1) $p' = p\cdot g_i+$ or $p' = p\cdot g_i-$ for some i, where i is a non-invariant if $p_1, p_2 \in \{1, 2\}$;

(2) $p' = p\cdot g_i+$, where $i = p_1$ if $p_1 \notin \{1, 2\}$;

(3) $p' = p\cdot g_i-$, where $i = p_2$ if $p_2 \notin \{1, 2\}$.

Repeat the above process until the path reaches the node I.

As an example, let $p = 14523 = (2\ 4)(5\ 3)$. A shortest path from p to I is as follows:

$$14523\ g_4-\ \ 21543\ g_3+\ \ 15243\ g_5-\ \ 31245\ g_3+\ \ 12345.$$

Let D_p denote the length measured in terms of the number of edges in the shortest path from p to I. We have the following lemma [8].

Lemma 2. If $p = c_1c_2\ldots c_ke_1e_2\ldots e_l \in A_n$, then

D_p	$= n+k-l$	if $p_1 = 1$ and $p_2 = 2$
	$= n+k-l-3$	if $p_1 = 2$ and $p_2 = 1$
	$= n+k-l-2$	if $p_1 \neq 1$ and $p_2 = 2$
	$= n+k-l-2$	if $p_1 = 1$ and $p_2 \neq 2$
	$= n+k-l-3$	if $1, 2 \in c_i$ for some i and $\mid c_i \mid \geq 3$
	$= n+k-l-4$	if $1 \in c_i$ and $2 \in c_j$, $i \neq j$.

The result of routing is obtained in the form of a product of generators. Since the product of two generators of alternating group graphs is not commutative, there is no trivial rule for finding alternative paths of the same length. However, like star graphs [6], rules for finding the alternative paths in an alternating group graph can be based on the concepts of "ordinary" and "barrel" products of generators. Ordinary and barrel products are basic parts of any shortest path that can be manipulated in order to obtain alternative paths. The whole analysis relies on the mapping of each generator to the corresponding movement of symbols. Therefore, the complex product can be tracked as a set of movements of symbols.

Let $\sigma(i)$ denote the *sign* + if i is even, or the sign $-$ if i is odd.

Definition 3. The *initial product* is the product of the following form: for some integer l,

$$h_1^{\sigma(l)} h_2^{\sigma(l+1)} \cdots h_n^{\sigma(l+n-1)}.$$

Definition 4. The *ordinary product* is any product that is an arbitrary permutation of an arbitrary subset of k distinct-index generators with alternating signs from the initial product.

Definition 5. The *barrel product* of generator is obtained when the first generator of an ordinary product is appended at the end of that product with alternating signs, i.e., the product of the following form:

$$h_1^{\sigma(i)} h_2^{\sigma(i+1)} \cdots h_n^{\sigma(i+n-1)}\ h_1^{\sigma(i+n)}.$$

The properties of the barrel product are exposed in the following lemma and its corollaries.

Lemma 6. For AG_n, the following holds. (Inverting signs)

$$g_3{+}g_4{-} \cdots g_{n-2}^{\sigma(n-3)} g_{n-1}^{\sigma(n-2)} g_n^{\sigma(n-1)} g_3^{\sigma(n)}$$
$$= g_3{-}g_4{+} \cdots g_{n-2}^{\sigma(n-2)} g_{n-1}^{\sigma(n-1)} g_n^{\sigma(n)} g_3^{\sigma(n+1)}.$$

Lemma 7. For AG_n, the following holds. (Rotation)

$$g_3{+}g_4{-}\cdots g_{n-2}^{\sigma(n-3)}g_{n-1}^{\sigma(n-2)}g_n^{\sigma(n-1)}g_3^{\sigma(n)}$$
$$= g_4{-}g_5{+}\cdots g_{n-1}^{\sigma(n-4)}g_n^{\sigma(n-3)}g_3^{\sigma(n-2)}g_4^{\sigma(n-1)}$$
$$\cdots$$
$$= g_{n-1}^{\sigma(n-2)}g_n^{\sigma(n-1)}g_3^{\sigma(n)}g_4^{\sigma(n+1)}\cdots g_{n-3}^{\sigma(2n-6)}g_{n-2}^{\sigma(2n-5)}g_{n-1}^{\sigma(2n-4)}.$$

Corollary 8. For AG_n, let $t_1, t_2, \ldots, t_m$ be distinct integers in $<n> - \{1, 2\}$. Then,

$$g_{t_1}{+}g_{t_2}{-}\cdots g_{t_m}^{\sigma(m-1)}g_{t_1}^{\sigma(m)}$$
$$= g_{t_1}{-}g_{t_2}{+}\cdots g_{t_m}^{\sigma(m)}g_{t_1}^{\sigma(m+1)}$$
$$= g_{t_2}{+}\cdots g_{t_m}^{\sigma(m-2)}g_{t_1}^{\sigma(m-1)}g_{t_2}^{\sigma(m)}$$
$$= \ldots$$
$$= g_{t_m}{+}g_{t_1}{-}\cdots g_{t_{m-1}}^{\sigma(m-1)}g_{t_m}^{\sigma(m)}.$$

That is, Lemmas 6 and 7 hold when an arbitrary ordinary product is substituted for the ordinary part of barrel product. A barrel product of k distinct-index generators can be represented in $2k$ different ways, all preserving the same cyclic ordering of generators.

Definition 9. Let $\prod = h_1^{\sigma(l)}h_2^{\sigma(l+1)}\cdots h_m^{\sigma(l+m-1)}$ be an ordinary or barrel product. The *invert-sign product* $\prod^-$ of $\prod$ is $h_1^{\sigma(l+1)}h_2^{\sigma(l+2)}\cdots h_m^{\sigma(l+m)}$. The *positive-sign product* $\prod^+$ of $\prod$ is $\prod$ itself.

The following two lemmas introduce the law of commutativity between the ordinary and barrel subproducts of generators.

Lemma 10. For AG_n, let $t_1, t_2, \ldots, t_m$ be distinct integers in $<n> - \{1, 2\}$. Then,

$$g_{t_1}{+}g_{t_2}{-}\cdots g_{t_j}^{\sigma(j-1)}g_{t_1}^{\sigma(j)}\boxed{g_{t_{j+1}}^{\sigma(j+1)}g_{t_{j+2}}^{\sigma(j+2)}\cdots g_{t_m}^{\sigma(m)}g_{t_{j+1}}^{\sigma(m+1)}}$$
$$= \boxed{g_{t_{j+1}}{+}g_{t_{j+2}}{-}\cdots g_{t_m}^{\sigma(m-j-1)}g_{t_{j+1}}^{\sigma(m-j)}}\,g_{t_1}^{\sigma(m-j+1)}g_{t_2}^{\sigma(m-j+2)}\cdots g_{t_j}^{\sigma(m)}g_{t_1}^{\sigma(m+1)}$$
$$= g_{t_1}{+}\,g_{t_2}{-}\cdots g_{t_i}^{\sigma(i-1)}\boxed{g_{t_{j+1}}^{\sigma(i)}g_{t_{j+2}}^{\sigma(i+1)}\cdots g_{t_m}^{\sigma(i+m-j-1)}g_{t_{j+1}}^{\sigma(i+m-j)}}\,g_{t_{i+1}}^{\sigma(i+m-j+1)}\cdots g_{t_j}^{\sigma(m)}g_{t_1}^{\sigma(m+1)}.$$

Lemma 11. For AG_n, let $t_1, t_2, \ldots, t_m$ be distinct integers in $<n> - \{1, 2\}$. Then,

$$g_{t_1}{+}g_{t_2}{-}\cdots g_{t_j}^{\sigma(j-1)}\boxed{g_{t_{j+1}}^{\sigma(j+1)}g_{t_{j+2}}^{\sigma(j+2)}\cdots g_{t_m}^{\sigma(m)}g_{t_{j+1}}^{\sigma(m+1)}}$$
$$= g_{t_1}{+}\,g_{t_2}{-}\cdots g_{t_i}^{\sigma(i-1)}\boxed{g_{t_{j+1}}^{\sigma(i)}g_{t_{j+2}}^{\sigma(i+1)}\cdots g_{t_m}^{\sigma(i+m-j-1)}g_{t_{j+1}}^{\sigma(i+m-j)}}\,g_{t_{i+1}}^{\sigma(i+m-j+1)}\cdots g_{t_j}^{\sigma(m)}$$
$$= \boxed{g_{t_{j+1}}^{\sigma(l)}g_{t_{j+2}}^{\sigma(l+1)}\cdots g_{t_m}^{\sigma(l+m-j-1)}g_{t_{j+1}}^{\sigma(l+m-j)}}\,g_{t_1}^{\sigma(m-j+1)}\,g_{t_2}^{\sigma(m-j+2)}\cdots g_{t_j}^{\sigma(m)}.$$

When commuting with a barrel product that contains odd number of generators, the ordinary product should turn to its invert-signed product. The following example will illustrate the nesting of subproducts.

$$\prod_o = \boxed{g_3{+}g_4{-}g_3{+}}g_5{+}g_6{-}$$
$$= g_5{-}g_6{+}\boxed{g_3{+}g_4{-}g_3{+}}$$
$$= g_5{-}g_6{+}\boxed{g_3{-}g_4{+}g_3{-}}$$
$$= g_5{-}\boxed{g_3{-}g_4{+}g_3{-}}g_6{-}$$
$$= g_5{-}\boxed{g_3{+}g_4{-}g_3{+}}g_6{-}.$$

Definition 12. The *coupled ordinary product* is a concatenation of two *ordinary products* such that all the generators are of distinct-index and their boundary is of the same sign, not alternating signs.

For example, $g_3{+}g_4{-}g_5{-}g_6{+}$ is a coupled ordinary product.

Lemma 13. Let $\prod_1$ and $\prod_2$ are two ordinary products. Suppose that $\prod_1\prod_2$ is a coupled ordinary product. Then,

$$\prod_1\prod_2 = \prod_2^{\sigma(l)}\prod_1^{\sigma(m)},$$

where l and m are the numbers of generators contained in $\prod_1$ and $\prod_2$, respectively.

Definition 14. The *generalized ordinary product* is either an ordinary product or a coupled ordinary product.

Corollary 15. If some product of generators in AG_n, consists of k products of distinct sets of generators

$$\prod_o = \prod_1\cdot\prod_2\cdots\prod_i\cdots\prod_k$$

and if at most one of these products is a generalized ordinary one while all other products are of barrel type, then the ordering of products $\prod_i$ in the overall product $\prod_o$ is arbitrary except that the signs of the generalized ordinary one may be obligated to invert.

Theorem 16. Every product of arbitrary number of generators in AG_n can be reduced to product of $k \geq 1$ subproducts of distinct sets of generators

$$\prod_m = \prod_1\cdot\prod_2\cdots\prod_i\cdots\prod_k$$

in which at most one subproduct is a generalized ordinary one and all others are of barrel type. Product $\prod_m$ cannot be further reduced, therefore it has the form of "minimal" product of generators.

Without loss of generality, we let $\prod_1$ be the generalized ordinary subproduct if it exists. Product $\prod_m$ will be referred to as minimal product and it actually represents the routing function in AG_n. The diameter of AG_n corresponds to the minimal product with the maximal number of generators. Since barrel product has the property of repeating the first generator, the maximal length of the minimal product corresponds to the maximal number of barrel products in it.

Lemma 17. The maximal number k of subproducts in a minimal product from Theorem 16 is $\lceil (n-2)/2 \rceil$.

3. FAULT DIAMETER

Let us consider two nodes A and B in AG_n and some shortest paths Π_{AB} between A and B. Suppose Π_{AB} $=\Pi_1\cdot\Pi_2\cdots\Pi_i\cdots\Pi_k$, which has determined subsets of generators and their relative ordering for each Π_i. In case no adjacent nodes or channels has failed, the message is free to leave the node along the first generator of the ordinary subpaths (with a proper sign), or along any of the generator that belongs to any barrel subpath from Π_{AB}, without increasing the length of the path. Now, suppose that $2(n-2)-1$ nodes that are adjacent to A have failed, and that the connection with the rest of the network is only along the channel corresponding to generator $g_o^{\sigma(l)}$, $3 \le o \le n$ and $l \in N$. In other words, the message is forced to leave the node A only along the channel $g_o^{\sigma(l)}$. Obviously, this tends to increase the path length between A and B. As a result or analysis, we shall compute worst cases of path lengthening, i.e., the fault diameter of AG_n. We distinguish the following cases:

(1) $g_o^{\sigma(l)}$ or $g_o^{\sigma(l+1)}$ is a generator contained in Π_{AB}. Then we have the following five subcases.

(1.a) One of the barrel subproducts contains $g_o^{\sigma(l)}$ or $g_o^{\sigma(l+1)}$. According to cummutativity of subproducts in the “minimal” product (shortest path), this subproduct can be put at the beginning of the path. Then according to the known features of barrel products, this subproduct can be transformed to its equivalent have $g_o^{\sigma(l)}$ at the beginning and at the end (with proper sign). Hence, a message can leave the node along $g_o^{\sigma(l)}$ without increasing the length of the path. Let us consider for example, path Π_{AB} = $g_3+g_4-g_5+g_6-g_5+$ and $g_o^{\sigma(l)}$ = g_6-. This path can be transformed in the following way:

$$\begin{aligned}\Pi_{AB} &= g_3+g_4-\ \boxed{g_5+g_6-g_5+}\\ &= \boxed{g_5+g_6-g_5+}\ g_3-g_4+\\ &= \boxed{g_6-g_5+\ g_6-}\ g_3-g_4+.\end{aligned}$$

(1.b) $g_o^{\sigma(l)}$ is at the beginning of the ordinary subpath. In this case, ordinary subpath can be placed at the beginning of the path, and message can again leave the node without lengthening the path. According to Lemma 13, we always assume that $g_o^{\sigma(l)}$ or $g_o^{\sigma(l+1)}$ is at the first ordinary subproduct of the generalized ordinary one without loss of generality.

(1.c) $g_o^{\sigma(l+1)}$ is at the beginning of the ordinary subpath. In this case the original shortest path must be lengthened and transformed, keeping that the permutation that maps A to B is not changed. This can be achieved by using the inverting property of generators:
If the ordinary product contains only one generator,

$$\begin{aligned}\Pi'_{AB} &= \Pi_1\cdot\Pi_2\cdots\Pi_k\\ &= g_o^{\sigma(l)}[\text{coup.}][\text{barr.}_1][\text{barr.}_2]\cdots[\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l+1)}g_o^{\sigma(l+1)}[\text{coup.}][\text{barr.}_1][\text{barr.}_2]\cdots[\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l+1)}[\text{barr.}_1]g_o^{\sigma(l+m+1)}[\text{coup.}]^{\sigma(m)}[\text{barr.}_2]\cdots[\text{barr.}_{k-1}],\end{aligned}$$

where m is the number of generators in the barrel product $[\text{barr.}_1]$. Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus one. In that case, the maximal length of Π_{AB} is d_n, $n \ge 5$. Thus, the maximal length of Π'_{AB} is d_n+1. It is trivial if there is no couple or barrel product:

$$\Pi'_{AB} = g_o^{\sigma(l+1)}g_o^{\sigma(l+1)}.$$

For example, $\Pi_{AB} = g_3+g_5+g_6-g_5+$ and $g_o^{\sigma(l)} = g_3-$. Then,

$$\begin{aligned}\Pi'_{AB} &= g_3-g_3-\boxed{g_5+g_6-g_5+}\\ &= g_3-g_5+g_6-g_5+g_3+.\end{aligned}$$

If there is only the couple product,

$$\begin{aligned}\Pi'_{AB} &= g_o^{\sigma(l+1)}\, g_o^{\sigma(l+1)}[\text{coup.}]\\ &= g_o^{\sigma(l+1)}\, g_o^{\sigma(l+1)}\ [g_u^{\sigma(l)}\ \text{coup.}']\\ &= g_o^{\sigma(l+1)}\, g_o^{\sigma(l+1)}\, g_u^{\sigma(l+1)}\, g_u^{\sigma(l+1)}\ [\text{coup.}']\\ &= g_o^{\sigma(l+1)}\, g_u^{\sigma(l)}\, g_o^{\sigma(l)}\, g_u^{\sigma(l+1)}\ [\text{coup.}'],\end{aligned}$$

where $g_u^{\sigma(l)}$ is the first generators in the couple product [coup.]. Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus two. In that case, the maximal length of Π_{AB} is $n-2$, $n \ge 4$. The maximal length of Π'_{AB} is n.
For example, $\Pi_{AB} = g_3+g_5+g_6-$ and $g_o^{\sigma(l)} = g_3-$. Then,

$$\begin{aligned}\Pi'_{AB} &= g_3-g_3-g_5+g_6-\\ &= g_3-g_3-g_5-g_5-g_6-\\ &= g_3-g_5+g_3+g_5-g_6-.\end{aligned}$$

Otherwise,

$$\begin{aligned}\Pi'_{AB} &= \Pi_1\cdot\Pi_2\cdots\Pi_k\\ &= [g_o^{\sigma(l)}\ \text{ord.}][\text{coup.}][\text{barr}_1][\text{barr}_2]\cdots[\text{barr.}_{k-1}]\\ &= \boxed{[g_o^{\sigma(l)}\ \text{ord.}]\ g_o^{\sigma(l+m+1)}}\ g_o^{\sigma(l+m)}\ [\text{coup.}]\ [\text{barr.}_1]\ [\text{barr.}_2]\ \cdots\\ &\quad [\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l+1)}\text{ord.}^{-}g_o^{\sigma(l+m)}g_o^{\sigma(l+m)}[\text{coup.}][\text{barr}_1][\text{barr}_2]\cdots[\text{barr}_{k-1}]\\ &= g_o^{\sigma(l+1)}\ \text{ord.}^{-}g_o^{\sigma(l+m+1)}[\text{coup.}][\text{barr.}_1][\text{barr.}_2]\cdots[\text{barr.}_{k-1}],\end{aligned}$$

where m is the number of generators in the ordinary product [ord.]. Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus one. The maximal length of Π'_{AB}

is d_n, $n \geq 4$. The maximal length of Π'_{AB} is $d_n + 1$. For example, $\Pi_{AB} = g_3{+}g_4{-}g_5{+}g_6{-}g_5{+}$ and $g_o^{\sigma(l)} = g_3{-}$. Then,

$$\begin{aligned}\Pi'_{AB} &= \boxed{g_3{+}g_4{-}g_3{+}} g_3{-}g_5{+}g_6{-}g_5{+}\\ &= g_3{-}g_4{+}g_3{-}g_3{-}g_5{+}g_6{-}g_5{+}\\ &= g_3{-}g_4{+}g_3{+}g_5{+}g_6{-}g_5{+}.\end{aligned}$$

(1.d) $g_o^{\sigma(l)}$ is contained in the ordinary subpath but not at the first position. The original shortest path must be lengthened and transformed.

$$\begin{aligned}\Pi'_{AB} &= [\text{ord.}' g_o^{\sigma(l)} \text{ord.}''][\text{coup.}][\text{barr.}_1][\text{barr.}_2]\cdots[\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l+m)} \boxed{g_o^{\sigma(l+m+1)}\ \text{ord.}'\ g_o^{\sigma(l)}}\ \text{ord.}''\ [\text{coup.}]\ [\text{barr.}_1]\\ &\quad [\text{barr.}_2]\cdots[\text{barr.}_{k-1}],\end{aligned}$$

where m is the number of generators in the ordinary product ord.′. The enclosed subproduct is now of barrel type. If m is even, $g_o^{\sigma(l+m)} = g_o^{\sigma(l)}$ and the enclosed subproduct can be rotated into form that begins with some generator from ord.′. Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus two. In that case, the maximal length of Π_{AB} is $3+\lfloor 3(n-5)/2 \rfloor$, $n \geq 5$. The maximal length of Π'_{AB} is $5+\lfloor 3(n-5)/2 \rfloor = 2+\lfloor 3(n-3)/2 \rfloor$. If m is odd, we can invert the sign of the enclosed subproduct as follows.

$$\begin{aligned}\Pi'_{AB} &= g_o^{\sigma(l+m)} \boxed{g_o^{\sigma(l+m)}\ \text{ord.}'^{-}\ g_o^{\sigma(l+1)}}\ \text{ord.}''\ [\text{coup.}]\ [\text{barr.}_1]\\ &\quad [\text{barr.}_2]\cdots[\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l)}\ \text{ord.}'^{-}\ g_o^{\sigma(l+1)}\ \text{ord.}''\ [\text{coup.}]\ [\text{barr.}_1]\ [\text{barr.}_2]\cdots\\ &\quad [\text{barr.}_{k-1}].\end{aligned}$$

Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus one. In that case, the maximal length of Π_{AB} is $2+\lfloor 3(n-4)/2 \rfloor$, $n \geq 4$. The maximal length of Π'_{AB} is $3+\lfloor 3(n-4)/2 \rfloor = \lfloor 3(n-2)/2 \rfloor = d_n$.

(1.e) $g_o^{\sigma(l+1)}$ is contained in the ordinary subpath but not at the first position. We can treat this case almost the same way to (1.d).

$$\begin{aligned}\Pi'_{AB} &= [\text{ord.}' g_o^{\sigma(l+1)} \text{ord.}''][\text{coup.}][\text{barr.}_1][\text{barr.}_2]\cdots[\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l+m+1)} \boxed{g_o^{\sigma(l+m+2)} \text{ord.}' g_o^{\sigma(l+1)}}\ \text{ord.}''\ [\text{coup.}]\ [\text{barr.}_1]\\ &\quad [\text{barr.}_2]\cdots[\text{barr.}_{k-1}],\end{aligned}$$

where m is the number of generators in the ordinary product ord.′. The enclosed subproduct is now of barrel type. If m is odd, $g_o^{\sigma(l+m+1)} = g_o^{\sigma(l)}$ and the enclosed subproduct can be rotated into form that begins with some generator from ord.′. Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus two. In that case, the maximal length of Π_{AB} is $2+\lfloor 3(n-4)/2 \rfloor$, $n \geq 4$. The maximal length of Π'_{AB} is $4+\lfloor 3(n-4)/2 \rfloor = 1+\lfloor 3(n-2)/2 \rfloor = d_n+1$. If m is even, we can invert the sign of the enclosed subproduct as follows.

$$\begin{aligned}\Pi'_{AB} &= g_o^{\sigma(l+m+1)} \boxed{g_o^{\sigma(l+m+1)}\ \text{ord.}'^{-}\ g_o^{\sigma(l)}}\ \text{ord.}''\ [\text{coup.}][\text{barr.}_1]\\ &\quad [\text{barr.}_2]\cdots[\text{barr.}_{k-1}]\\ &= g_o^{\sigma(l)}\ \text{ord.}'^{-}\ g_o^{\sigma(l)}\ \text{ord.}''\ [\text{coup.}]\ [\text{barr.}_1]\ [\text{barr.}_2]\cdots\\ &\quad [\text{barr.}_{k-1}].\end{aligned}$$

Therefore, the length of Π'_{AB} is no more than that of Π_{AB} plus one. In that case, the maximal length of Π_{AB} is $3+\lfloor 3(n-5)/2 \rfloor$, $n \geq 5$. The maximal length of Π'_{AB} is thus $4+\lfloor 3(n-5)/2 \rfloor = 1+\lfloor 3(n-3)/2 \rfloor$.

(2) Neither $g_o^{\sigma(l)}$ nor $g_o^{\sigma(l+1)}$ is a generator contained in Π_{AB}. Then we have two different cases.

(2.a) If Π_{AB} conatins at least one barrel subproduct. (It may optionally contain ordinary subproduct but this does not affect the analysis). In that case we again use inverting property of generators to lengthen the path without changing the mapping between A and B.

$$\Pi'_{AB} = g_o^{\sigma(l)}\ g_o^{\sigma(l+1)}\ [\text{ord.}\ ][\text{coup.}][\text{barr.}].$$

Since $g_o^{\sigma(l+1)}$ is not contained in subproducts in Π_{AB}, one barrel product can be nested between $g_o^{\sigma(l)}\ g_o^{\sigma(l+1)}$ and Π'_{AB} is given the following form:

$$\begin{aligned}\Pi'_{AB} &= g_o^{\sigma(l)}\ g_o^{\sigma(l+1)}[\text{ord.}][\text{barr.}][\text{coup.}]^{\sigma(m)}\\ &= g_o^{\sigma(l)}\ g_o^{\sigma(l+1)}\ [\text{barr.}][\text{ord.}]^{\sigma(m)}[\text{coup.}]^{\sigma(m)}\\ &= g_o^{\sigma(l)}\ [\text{barr.}]\ g_o^{\sigma(m+l+1)}\ [\text{ord.}]^{\sigma(m)}\ [\text{coup.}]^{\sigma(m)},\end{aligned}$$

where m is the number of generators in the barrel product [barr.]. The maximal length of Π'_{AB} is $d_{n-1}+2$. This value is also a candidate for fault diameter since $d_{n-1}+2 \geq d_n$. For example, $\Pi_{AB} = g_3{+}g_5{+}g_6{-}g_5{+}$ and $g_o^{\sigma(l)} = g_4{-}$. Then, Π'_{AB} is obtained in the following way:

$$\begin{aligned}\Pi'_{AB} &= g_4{-}g_4{+}g_3{+}\boxed{g_5{+}g_6{-}g_5{+}}\\ &= g_4{-}g_4{+}\boxed{g_5{+}g_6{-}g_5{+}}\ g_3{-}\\ &= g_4{-}\boxed{g_5{+}g_6{-}g_5{+}}\ g_4{-}g_3{-}.\end{aligned}$$

(2.b) If Π_{AB} contains only a generalized ordinary subproduct. In that case we must append a pair of $g_o^{\sigma(l)}$ and $g_o^{\sigma(l+1)}$ at the beginning and at the end of Π_{AB} in order to form a barrel product. If the number of generators in the ordinary product is odd,

$$\begin{aligned}\Pi'_{AB} &= g_o^{\sigma(l+1)}\ g_o^{\sigma(l)}\ [\text{ord.}]\ g_o^{\sigma(l)}\ g_o^{\sigma(l+1)}\ [\text{coup.}]\\ &= g_o^{\sigma(l+1)}\ g_o^{\sigma(l+1)}\ [\text{ord.}]^{-}\ g_o^{\sigma(l+1)}\ g_o^{\sigma(l+1)}\ [\text{coup.}]\\ &= g_o^{\sigma(l)}\ [\text{ord.}]^{-}\ g_o^{\sigma(l)}\ [\text{coup.}].\end{aligned}$$

If the number of generators in the ordinary subproduct is even,

$\Pi'_{AB} = g_o^{\sigma(l+1)}\, g_o^{\sigma(l)}$ [ord.] $g_o^{\sigma(l+1)}\, g_o^{\sigma(l)}$ [coup.]
$= g_o^{\sigma(l+1)}\, g_o^{\sigma(l+1)}$ [ord.]$^-$ $g_o^{\sigma(l)}\, g_o^{\sigma(l)}$ [coup.]
$= g_o^{\sigma(l)}$ [ord.]$^-$ $g_o^{\sigma(l+1)}$ [coup.].

Evidently, this path transformation increases its length by 2. The maximum length of Π'_{AB} is $n-3+2=n-1$.

4. CONCLUDING REMARKS

Since d_n is not a linear function of n, it seems convenient to determine the fault diameter by representing the candidates in tabular form, as shown in Table 1. From Table 1, we can conclude that $d_n^f = d_n+1$ for AG_n, i.e., the fault diameter of the alternating graphs is optimal.

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Table 1. The maximum lengths of Π'_{AB} and the fault diameters d^f.

AG_n	(1.a) (1.b)	(1.c)	(1.d) m even	(1.d) m odd	(1.e) m odd	(1.e) m even	(2.a)	(2b)	d^f
n	d_n	d_n+1	$\lfloor 3(n-3)/2 \rfloor+2$	$\lfloor 3(n-2)/2 \rfloor$	$\lfloor 3(n-2)/2 \rfloor+1$	$\lfloor 3(n-3)/2 \rfloor+1$	$d_{n-1}+2$	$n-1$	d_n+1
3	1	2	–	–	–	–	–	–	2
4	3	4	–	3	4	–	3	3	4
5	4	5	5	4	5	4	5	4	5
6	6	7	6	6	7	5	6	5	7
7	7	8	8	7	8	7	8	6	8
8	9	10	9	9	10	8	9	7	10
9	10	11	11	10	11	10	11	8	11
10	12	13	12	12	13	11	12	9	13
11	13	14	14	13	14	13	14	10	14
12	15	16	15	15	16	14	15	11	16