Probe Bipartite Distance-Hereditary Graphs

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***Abstract*—A graph $G = (V, E)$ is called a *probe graph* of graph class $\mathcal{G}$ if V can be partitioned into two sets $\mathbb{P}$ (probes) and $\mathbb{N}$ (nonprobes), where $\mathbb{N}$ is an independent set, such that G can be embedded into a graph of $\mathcal{G}$ by adding edges between certain nonprobes. A graph is *distance hereditary* if the distance between any two vertices remains the same in every connected induced subgraph. Bipartite distance-hereditary graphs are both bipartite and distance hereditary. In this paper we give an $O(nm)$-time algorithm for recognizing probe graphs of bipartite distance-hereditary graphs.**

I. Introduction

A *probe graph* P is a two-tuple (G, L) where G is a graph and L is a function from V_G to the set $\{\mathbb{P}, \mathbb{N}, \mathbb{U}\}$ of labels. We use P_G and P_L for the first and second tuple of P, respectively, and use P_V and P_E for the sets of vertices and edges of P_G, respectively. We also use $P_\mathbb{P}$, $P_\mathbb{N}$, and $P_\mathbb{U}$ for the sets of vertices $v \in P_V$ with $P_L(v) = \mathbb{P}$, $P_L(v) = \mathbb{N}$, and $P_L(v) = \mathbb{U}$, respectively. A probe graph P is *fully* (*resp. partially*) *partitioned* if $P_\mathbb{U} = \emptyset$ (resp. $P_\mathbb{U} \neq \emptyset$). A probe graph P is *unpartitioned* if $P_\mathbb{P} = P_\mathbb{N} = \emptyset$. We call probe graph P' a *subgraph* of a probe graph P if P'_G is a subgraph of P_G and $P'_L(v) = P_L(v)$ for $v \in P'_V$. Let X be a subset of P_V. A *subgraph of P induced by X* is the subgraph P' of P with $P'_G = P_G[X]$, i.e., P'_G is the subgraph of P_G induced by X. For $v \in P_V$, use $P - v$ to denote the probe subgraph of P induced by $P_V - v$. We also use $P - X$ for the subgraph of probe graph P induced by $P_V - X$ for a subset X of P_V. We call a vertex $v \in P_V$ a *probe*, a *nonprobe*, and a *prime* if $P_L(v) = \mathbb{P}$, $P_L(v) = \mathbb{N}$, and $P_L(v) = \mathbb{U}$, respectively.

A probe graph P is *feasible* if $P_\mathbb{N}$ is an independent set of P_G. We say a probe graph P^* is an *embedding* of probe graph P if $P^*_V = P_V$, $P_E \subseteq P^*_E$, $P_\mathbb{N} \subseteq P^*_\mathbb{N}$, $P_\mathbb{P} \subseteq P^*_\mathbb{P}$, P^* is fully partitioned, i.e., $P^*_\mathbb{U} = \emptyset$, $P^*_\mathbb{N}$ is an independent set of P_G, and for $(u, v) \in P^*_E - P_E$ we have $P^*_L(u) = P^*_L(v) = \mathbb{N}$. Let $\mathcal{G}$ be a class of graphs. We call probe graph P a *probe $\mathcal{G}$ graph* if there exists an embedding P^* of P such that $P^*_G \in \mathcal{G}$. If probe graph P is not feasible, then it does not have any embedding by definition and hence it is not a probe $\mathcal{G}$ graph for any graph class $\mathcal{G}$. The recognition of fully partitioned (resp. unpartitioned) probe $\mathcal{G}$ graphs is to determine whether a fully partitioned (resp. unpartitioned) probe graph has an embedding in $\mathcal{G}$.

The concept of probe graphs starts in 1989 [17]. Hertz introduced *slim graphs*, the probe graph of *Meyneil graphs*, and showed that slim graphs are perfect. Later Zhang *et al.* [22] introduced the recognition of fully partitioned probe interval graphs for solving the problem called *physical mapping* with connection of the human genome project. Since then the recognition of probe graphs of different graph classes appeals to many researchers [2], [4]–[11], [15], [18], [20], [21]. For the list of results on the recognition of probe graphs in different graph classes, please refer to Table 1 in [12].

The recognition of fully partitioned probe $\mathcal{G}$ graphs is a special case of the graph sandwich problem [14]. Given $G^1 = (V, E^1)$ and

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$G^2 = (V, E^2)$ where $E^1 \subseteq E^2$, the *graph sandwich problem* asks whether there exists a graph $G = (V, E)$, $E^1 \subseteq E \subseteq E^2$, where G is in a specific graph class $\mathcal{G}$. For example, the interval sandwich problem asks "Is there an interval graph $G = (V, E)$ where $E^1 \subseteq E \subseteq E^2$?". The partitioned probe graph recognition problem is equivalent to the graph sandwich problem in which $G^1 = G$ and $E^2 = E^1 + \{(u, v) \mid P_L(u) = P_L(v) = \mathbb{N}\}$.

Instead of studying the recognition of fully partitioned (or unpartitioned) probe $\mathcal{G}$ graphs directly, we study the recognition of *partially partitioned probe $\mathcal{G}$ graphs*. The recognition of partially partitioned probe $\mathcal{G}$ graphs is equivalent to the recognition of fully partitioned $\mathcal{G}$ graphs if $P_{\mathbb{U}} = \emptyset$ and is equivalent to the recognition of unpartitioned probe $\mathcal{G}$ graphs if $P_{\mathbb{P}} = P_{\mathbb{N}} = \emptyset$. In this paper, we give an $O(nm)$-time algorithm to recognize *partially partitioned probe bipartite distance-hereditary graphs*.

II. Preliminaries

For a vertex v of G, the *open neighborhood* of v, denoted by $N_G(v)$, consists of all vertices adjacent to v in G. We use $N_G[v]$ for $N_G(v) + v$, called the *closed neighborhood* of v. For a subset X of V, we use $N_G(X) = \cup_{x \in X} N_G(x) - X$ to denote the neighborhood of X in G. A subset X of V is called a *module* in G if for every $x \in X$ $N_G(x) - X = N_G(X)$. Two vertices $u \neq v$ are *false twins* in G if $N_G(u) = N_G(v)$ and are *true twins* if $N_G[u] = N_G[v]$. We say they are twins if $N_G(u) - v = N_G(v) - u$. A vertex v in G is called a *pendant vertex* if the degree of v is one. A vertex v in G is called a *universal vertex* if the degree of v is $|V| - 1$. In a graph $G = (V, E)$, two disjoint subsets S and T of V are *fully adjacent* if every vertex of S is adjacent to all vertices in T. Two sets A and B are *incomparable* if $A \cap B \neq \emptyset$, $A - B \neq \emptyset$, and $B - A \neq \emptyset$. For two vertices $u, v \in V$, we use $d_G(u, v)$ to denote the distance of u and v in a graph $G = (V, E)$.

We say a graph G is a *distance-hereditary graph* (**DHG** for short) if the distance between any two vertices remains the same in every connected induced subgraph of G. It is a classical result that distance-hereditary graphs can be captured by forbidden induced subgraphs [1]. For the house, hole, domino, and gem, we refer to Fig. 1. A hole is a k-cycle where $k \geq 5$.

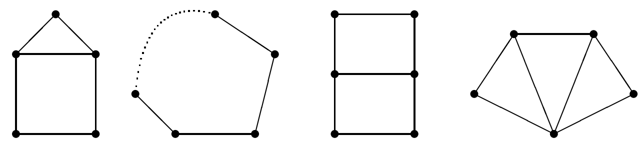

Figure 1: A house, a hole, a domino, and a gem.

Theorem 1. [1] *Let G be a graph. The following conditions are equivalent:*
1) *G is distance hereditary.*
2) *G contains no house, hole, domino, or gem as an induced subgraph.*
3) *Every connected induced subgraph of G with at least two vertices has a pendant vertex or a twin.*
4) *For every pair of vertices x and y with $d(x, y) = 2$, there is no induced x, y-path of length greater than 2.*

Corollary 1. *A bipartite graph G is a bipartite distance-hereditary graph if and only if it has no induced domino nor induced hole.*

Corollary 2. *A graph G is a bipartite distance-hereditary graph if and only if every connected induced subgraph of G with at least two vertices has a pendant vertex or a false twin.*

Definition 1. [13] *The hanging Φ of $G = (V, E)$ by v is an $(\ell + 1)$-tuple $(L_0, L_1, \ldots, L_\ell)$ where $\ell = \max_{u \in V} d_G(u, v)$, $L_0 = \{v\}$, and $L_i = \{u \in V \mid d_G(u, v) = i\}$ for $1 \leq i \leq \ell$.*

Definition 2. *Let $\Phi = (L_0, L_1, \ldots, L_\ell)$ be a hanging of G. For $x \in L_i$, $0 < i \leq \ell$, use $N_\Phi^-(x)$ for $N_G(x) \cap L_{i-1}$. Denote the subgraph of G induced by $\cup_{i \leq j \leq \ell} L_j$ by G_i for $0 \leq i \leq \ell$. By definition, $G = G_0$. Let x and y be vertices in L_i with $1 \leq i \leq \ell$. We say that (i) x* properly contains *y, denoted by $x \gg y$, if $N_\Phi^-(x)$ properly contains $N_\Phi^-(y)$; (ii) x and y are* equivalent, *denoted by $x \equiv y$, if $N_\Phi^-(x) = N_\Phi^-(y)$; and (iii) x is* minimal *(resp.* maximal*) if there does not exist any other vertex $z \in L_i$ such that $x \gg z$ (resp. $z \gg x$).*

Remark 1. *Let C be a component of G_i where $0 < i \leq \ell$. By definition of hanging, $N_G(C) \subseteq L_{i-1}$.*

Theorem 2. [13] *A connected graph G is distance hereditary if and only if for every hanging $\Phi = (L_0, L_1, \ldots, L_\ell)$ of G and every pair of vertices $x, y \in L_i$ ($1 \leq i \leq \ell$) that are in the same component of G_i, we have $N_\Phi^-(x) = N_\Phi^-(y)$. In other words, for a component C of G_i, $N_G(C)$ and $C \cap L_i$ are fully adjacent.*

Theorem 3. [16] *Suppose $\Phi = (L_0, L_1, \ldots, L_\ell)$ is a hanging of a connected distance-hereditary graph G. For any two vertices $x, y \in L_i$ with $1 \leq i \leq \ell$, $N_\Phi^-(x)$ and $N_\Phi^-(y)$ are disjoint or $N_\Phi^-(x) \subseteq N_\Phi^-(y)$ or $N_\Phi^-(y) \subseteq N_\Phi^-(x)$.*

The following corollary can be seen from the above two theorems.

Corollary 3. *Suppose $\Phi = (L_0, L_1, \ldots, L_\ell)$ is a hanging of a connected bipartite distance-hereditary graph G. $L_0, L_1, \ldots, L_\ell$ are independent sets of G. For any two components C_1 and C_2 of G_i with $1 \leq i \leq \ell$, $N_G(C_1)$ and $N_G(C_2)$ are disjoint or $N_G(C_1) \subseteq N_G(C_2)$ or $N_G(C_2) \subseteq N_G(C_1)$.*

Theorem 4. [16] *Suppose $\Phi = (L_0, L_1, \ldots, L_\ell)$ is a hanging of a connected distance-hereditary graph G. For each $1 \leq i \leq \ell$, there exists a minimal vertex v. In addition, if v is minimal then $N_G(x) - N_\Phi^-(v) = N_G(y) - N_\Phi^-(v)$ for every pair of vertices x and y in $N_\Phi^-(v)$.*

By Theorem 2 and 4, we get the corollary.

Corollary 4. *Suppose $\Phi = (L_0, L_1, \ldots, L_\ell)$ is a hanging of a connected bipartite distance-hereditary graph G. For each $1 \leq i \leq \ell$, G_i has a minimal component C, i.e., $N_G(C)$ does not properly contain $N_G(C')$ for any component C' of G_i. In addition, if C is a minimal component of G_i then $N_G(C)$ is a module of G.*

Corollary 5. *Suppose G is a biconnected bipartite distance-hereditary graph and $\Phi = (L_0, L_1, \ldots, L_\ell)$ is a hanging of G. Let C be a component of G_i where $1 < i \leq \ell$. $N_G(C)$ contains two vertices that are false twins in G.*

Proof: For every component C of G_i, there exists a minimal component C^* such that $N_G(C^*) \subseteq N_G(C)$. For every component C of G_i, $N_G(C)$ is an independent set of G. Because G is biconnected, $|N_G(C^*)| > 1$. By Corollary 4, $N_G(C^*)$ is a module of G. Since $N_G(C^*)$ is an independent set of G, any two vertices in $N_G(C^*)$ are false twins in G. ■

III. Some characteristics of probe bipartite distance-hereditary graphs

In this section, we give some observations on a probe bipartite distance-hereditary graph and its bipartite distance-hereditary embedding.

Lemma 1. *Probe bipartite distance-hereditary graphs are hole-free.*

Proof: Assume that $P = (P_G, P_L)$ is a probe bipartite distance-hereditary graph and $P^* = (P_G^*, P_L^*)$ is an embedding of P. Notice that P_G is a bipartite graph, there is no induced odd hole in P_G. Assume that P_G is not hole-free, i.e., there is a large even hole in G with length greater than or equal to six.
Let $C_k = v_1 v_2 \ldots v_k v_1$, $k \geq 6$, be an even hole of length k in P_G. Suppose that in P_G there are two adjacent vertices in C_k which are probes in P^*. If there are no consecutive probes in $P^*[C_k]$, we have alternating probes and nonprobes in C_k. It means all nonprobes are in the same color class of G. We can not add any edge between nonprobes to destroy C_k. Without loss of generality, assume that v_1 and v_2 are two consecutive probes in C_k. Let $\mathfrak{P}$ be a shortest path from v_3 to v_k in P_G with internal vertices in $C_k - \{v_1, v_2, v_3, v_k\}$. If $|\mathfrak{P}| > 3$, then $v_1 v_2 \mathfrak{P} v_1$ is a cycle of length greater than or equal to six in P_G^*. It contradicts the assumption that P^* is an embedding of P. If $|\mathfrak{P}| = 2$, v_3 and v_k must be nonprobes; v_4 and v_{k-1} must be probes. Let $\mathfrak{P}'$ be a shortest path from v_3 to v_k, $\mathfrak{P}' \neq \mathfrak{P}$, with internal vertices in $C_k - \{v_1, v_2, v_3, v_k\}$. If $|\mathfrak{P}'| = 4$ with $\mathfrak{P}' = v_3 v_i v_j v_k$, then $\{v_1, v_2, v_3, v_i, v_j, v_k\}$ induces a domino in P_G. If $|\mathfrak{P}'| > 4$, it implies $\mathfrak{P}' \geq 6$. $\mathfrak{P}' v_3$ forms a hole of length greater than or equal to six, where $\mathfrak{P}' = v_3 \ldots v_k$. It contradicts the assumption that P^* is an embedding of P. Therefore, probe bipartite distance-hereditary graphs are hole-free. ■

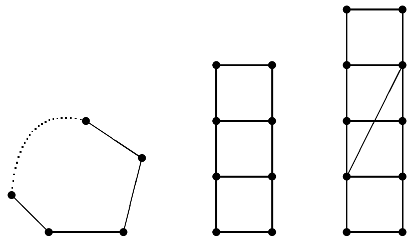

Figure 2: Some forbidden subgraphs of probe bipartite distance-hereditary graphs.

Lemma 2. *If P is probe bipartite distance-hereditary and the subgraph of G induced by a set D of six vertices is a domino, then D has exactly two nonprobes which are at distance three and the two vertices of degree three in the subgraph of P_G induced by D are probes.*

Proof: It can be shown by analyzing all cases. ∎

In Fig. 2, we list three forbidden subgraphs of probe bipartite distance-hereditary graphs. By Lemma 1, we see that Fig. 2(a) is a forbidden subgraph of probe bipartite distance-hereditary graphs. By Lemma 2, both vertices of degree three in an induced domino are probes. In Fig. 2(b), the four vertices of degree three must be probes, it is impossible to add any edge to destroy the induced dominos in it. In Fig 2(c), let the five vertices in the left-hand side from the bottom up be u_1, u_2, u_3, u_4, u_5, let the other five vertices in the right-hand side from the bottom up be v_1, v_2, v_3, v_4, v_5. Since the vertices of degree three in any induced domino are probes, u_2, v_2, u_4, and v_4 are probes. Since $\{u_3, u_2, u_4, v_4, u_5, v_5\}$ induces a domino, u_4 and v_4 are of degree three and u_2 is a probe, u_3 and v_5 must be nonprobes. Also $\{u_1, v_1, u_2, v_2, v_4, v_3\}$ induces a domino, u_2 and v_2 are of degree three and v_4 is a probe, v_3 and v_1 must be nonprobes. We see that (u_3, v_3) is an edge in the graph, but they have to be labeled as nonprobes in any probe bipartite distance-hereditary embedding, a contradiction. This shows that Fig. 2(c) is also a forbidden subgraph of probe bipartite distance-hereditary graphs.

Proposition 1. *Suppose P is a probe bipartite distance-hereditary graph and P^* is a bipartite distance-hereditary embedding of P. Then the following statements are true:*

1) *Any two probes in P^* that are false twins in P^* are false twins in P.*
2) *Any two nonprobes in P^* that are false twins in P^* are false twins in P.*

Definition 3. *Two disjoint vertex sets X and Y are called* probe adjacent *if X can be partitioned into two non-empty sets X_1 and X_2 and Y can be partitioned into two non-empty sets Y_1 and Y_2 such that every vertex in X_1 (resp. Y_1) is adjacent to all vertices of Y (resp. X) and every vertex in X_2 (resp. Y_2) is adjacent to all vertices of Y_1 (resp. X_1) but not adjacent to any vertex of Y_2 (resp. X_2).*

According the definition of probe adjacent, we have the following lemma.

Lemma 3. *Let P be a probe graph and P^* be a bipartite distance-hereditary embedding of P. Suppose X and Y are two disjoint vertex sets of P_G of size greater than one and X and Y are fully adjacent in P^*_G. If both X and Y have vertices with labels both $\mathbb{P}$ and $\mathbb{N}$ in P^*, then X and Y are probe adjacent in G. Besides, a vertex $x \in X$ (resp. Y) is a probe in P^* if and only if x is adjacent to all vertices in Y (resp. X).*

IV. The algorithm

In this section, we give an $O(nm)$-time algorithm to recognize probe bipartite distance-hereditary graphs. This algorithm is a recursive one. We denote the input probe graph by P. The algorithm first check whether P is a bipartite graph. If P is not bipartite, then it has no bipartite distance-hereditary embedding. If P is bipartite, the algorithm checks whether P is feasible. If P is not feasible, then it is not a probe bipartite distance-hereditary graph. Set $P_L(u) = \mathbb{P}$ for all vertices of $u \in P_{\mathbb{U}}$ that is adjacent to $v \in P_V$ with $P_L(v) = \mathbb{N}$. This can be done in linear time. In the following assume P is bipartite and feasible, i.e., all neighbors of a nonprobe must be probes. The algorithm checks to which of the following classes the input probe graph P does belong and takes action accordingly:

P 1 P_G has false twins. If it has one, reduce the size of P according to Corollary 6 and solve the problem recursively. The corollary and the reduction steps will be described in Section IV-A.

P 2 $|P_V| \leq c$ for some constant c. Solve the problem by brute force in $O(1)$ time.

P 3 P is fully partitioned. Use the $O(n^2)$-time algorithm in [4].

P 4 P_G is biconnected and without false twins. Call Algorithm **B**, to be given in Section IV-B,

P 5 P_G is not biconnected and without false twins. Call Algorithm **R**, to be given in Section IV-C to solve the problem recursively.

It is easy to see the correctness of the algorithm if the algorithm for each class of input is correct. We analyze the time complexity of the algorithm in Section IV-D.

A. False twins

In this subsection we first prove a lemma and show how to use it to solve the problem recursively.

Theorem 5. [12] *Suppose P is a probe graph and u and v are false twins in P_G satisfying one of the following conditions.*

1) $P_L(u) = P_L(v) = \mathbb{P}$, $\mathbb{N}$, *or* $\mathbb{U}$.
2) $P_L(u) = \mathbb{P}$, $P_L(v) = \mathbb{N}$ *or* $\mathbb{U}$.
3) $P_L(u) = \mathbb{N}$, $P_L(v) = \mathbb{U}$.

Then P is a probe distance-hereditary graph if and only if $P - v$ is a probe distance-hereditary graph.

Corollary 6. *Suppose P is a probe graph and u and v are false twins in P_G satisfying one of the three conditions of Theorem 5. Then P is a probe bipartite distance-hereditary graph if and only if $P - v$ is a probe bipartite distance-hereditary graph.*

Proof: If P has a bipartite distance-hereditary embedding P^*, then $P^* - v$ is a bipartite distance-hereditary embedding of $P - v$. Next we show that P has a bipartite distance-hereditary embedding if $P - v$ has one. Suppose $P - v$ has a bipartite distance-hereditary embedding P'. We then obtain P^* from P' by attaching v as a false twin of u. Let $P^*_L(v) = P'_L(u)$ if $P_L(v) = \mathbb{U}$. If $P^*_L(u) = \mathbb{P}$, we see $N_P(u) = N_{P^*}(u)$ and $N_P(v) = N_{P^*}(v)$. Suppose $P^*_L(u) = \mathbb{N}$ and $P_L(v) = \mathbb{N}$ or $\mathbb{U}$. Assume $N_{P^*}(u) = N_P(u) + X$, all vertices in X are nonprobes. We obtain P^* from P' by attaching v as a false twin of u and letting $P^*_L(v) = \mathbb{N}$ if $P_L(v) = \mathbb{U}$. Hence $N_{P^*}(v) = N_{P^*}(u) = N_P(u) + X$. By Corollary 2, we see P^* is a bipartite distance-hereditary embedding of P. ■

The proof of the above corollary explicitly points out how to reduce the size of input probe graph P and implies the problem can be solved recursively.

In [19], an $O(n^2)$-time algorithm was developed to remove all false twins in a given graph. We end the subsection by the following lemma.

Lemma 4. *Given a graph, removing vertices that have a false twin until it is not possible can be done in $O(n^2)$ time.*

B. Kernel probe graphs and Algorithm B

In the subsection we deal with the case that input graph P is of class **P 4**. This is the most crucial part of the algorithm. We will show that whether such a probe graph P is a probe bipartite distance-hereditary graph can be recognized in $O(n^2)$ time. First arbitrarily pick an edge (x, y) of P_E. In any bipartite distance-hereditary embedding of P, either x is a probe or y is a probe. Hence we reduce the problem to the case that there is a vertex $p \in P_V$ with $P_L(p) = \mathbb{P}$. We call a probe graph P satisfying the following three conditions a *kernel probe graph*: (i) P_G is biconnected, (ii) P_G has no false twins, and (iii) there is a vertex $p \in P_V$ with $P_L(p) = \mathbb{P}$. Given a kernel probe graph P and a probe p, our goal is to determine whether P is a probe bipartite distance-hereditary graph. We say that a kernel probe graph P is *well-labeled* if there is a vertex p such that $P_L(p) = \mathbb{P}$ and $P_L(x) \neq \mathbb{U}$ for every vertex x in the open neighborhood of p in P_G. Let $\Phi = (L_0, L_1, \ldots, L_\ell)$ be the hanging of P_G by p. Since $L_0, L_1, \ldots, L_\ell$ are independent sets of P_G

and P_G are biconnected, $\ell > 1$. For clarity of the notation, use G for P_G. Algorithm **B** checks to which class of probe graphs the input kernel probe graph P does belong and takes action accordingly:

C 1. P is well-labeled.

C 2. P is not well-labeled.

To handle the case that P is of class **C 2**, we have the following three subcases.

D 1. $\ell > 2$.

D 2. $\ell = 2$ and there is a vertex $q \in L_2$ with $P_L(q) = \mathbb{P}$.

D 3. $\ell = 2$ and every vertex in L_2 is not a probe.

In the following assume that P is a kernel probe graph and P^* is a minimal bipartite distance-hereditary embedding of P. For simplicity, use G and G^* for P_G and P_G^*, respectively. Let p be a probe of P^*, $\Phi = (L_0, L_1, \ldots, L_\ell)$ and $\Psi = (Z_0, Z_1, \ldots, Z_h)$ be the hangings of G and G^* by vertex p, respectively. The above notation will be used in lemmas and theorems in the rest of this subsection. Now we give some observations on both the hangings of G and G^* by a probe p. Since P^* is also a distance-hereditary embedding of P, P^* has all properties of the distance-hereditary embedding of P that are shown in [12].

Theorem 6. [12] *Suppose $P = (G, P_L)$ is a probe bipartite distance-hereditary graph and $P^* = (G^*, P_L^*)$ is a minimal embedding of P. Let p be a probe of P^*, $\Phi = (L_0, L_1, \ldots, L_\ell)$ and $\Psi = (Z_0, Z_1, \ldots, Z_h)$ be the hangings of G and G^* by vertex p, respectively. Then the following statements hold.*

1) *Suppose C is a component of G_i^* with $1 < i \leq h$. Then $N_{G^*}(C)$ contains probes and nonprobes in P^*. In addition, if $|C \cap Z_i| > 1$ then $C \cap Z_i$ also contains probes and nonprobes in P^*.*
2) *For $x \in Z_i$ where $1 < i \leq h$, $N_\Psi^-(x)$ contains probes and nonprobes in P^*.*
3) *$\ell = h > 1$ and $L_i = Z_i$ for $0 \leq i \leq \ell = h$.*
4) *For $1 < i \leq \ell$ and $x \in L_i$, x is a probe in P^* if and only if in G x is adjacent to some vertices in L_{i-1} that are nonprobes in P^*.*

Algorithm W

Now we are ready to show the algorithm for the case that P is of class **C 1**, i.e., a well-labeled kernel probe graph. We refer to the algorithm for handling this case as **Algorithm W**. We will see that Algorithm **W** serves as a major subroutine to be used later. The algorithm is as follows. By definition, the labels of vertices in $N_G[p]$ are either $\mathbb{P}$ or $\mathbb{N}$. Compute P' from P as follows. Let $P'_G = P_G$ and let $P'_L(y) = P_L(y)$ for all $y \in N_G[p]$. For every i from $i = 2$ to $i = \ell$ and every $y \in L_i$ with $P_L(y) = \mathbb{U}$, let $P'_L(y) = \mathbb{P}$ if in G y is adjacent to some vertex $z \in L_{i-1}$ with $P'_L(z) = \mathbb{N}$; and let $P'_L(y) = \mathbb{N}$ otherwise. By Theorem 6, we see that P is a probe bipartite distance-hereditary graph if and only if P' is a probe bipartite distance-hereditary graph. Apparently P' is fully partitioned. Use the $O(n^2)$-time algorithm in [4] to determine whether P' is a probe bipartite distance-hereditary graph. It is not hard to see that Algorithm **W** runs in $O(n^2)$ time.

In the following we give observations to be used for handling probe graphs of class **C 2**.

Lemma 5. *Suppose P^* is a minimal bipartite distance-hereditary embedding of P. Then the following statements hold:*

1) *A component of G_i is a component of G_i^* for $0 \leq i \leq \ell$.*
2) *For any component C of G_i with $1 \leq i \leq \ell$, $C \cap L_i = C \cap Z_i$.*
3) *For any component C of G_i with $1 \leq i \leq \ell$ and $|C \cap L_i| > 1$, $N_G(C) = N_{G^*}(C)$.*

Proof: First we prove Statement (1). By Theorem 6, $h = \ell$ and $L_i = Z_i$ for $0 \leq i \leq h = \ell$. In addition, G^* is obtained from G by adding edges. Hence a component of G_i^*, $0 \leq i \leq h$, is a component of G_i or the union of some components of G_i. Since both G and G^* are biconnected, all G_0, G_1, G_0^*, and G_1^* have only one component. Hence the lemma holds for $i = 0$ and $i = 1$. For $1 < i \leq \ell$, we prove the statement by contradiction showing that if some component C of G_i^* is not a component of G_i then P^* is not a minimal bipartite distance-hereditary embedding of P. Suppose C is a component of G_i^* that properly contains a component D

of G_i. Let P' be an embedding of P obtained from P^* by removing edges connecting a vertex in $C - D$ and another vertex in D. Use G' for P'_G. Clearly $N_{G^*}(C) = N_{G^*}(D) \cap Z_{i-1} = N_{G^*}(C - D) \cap Z_{i-1} = N_{G'}(D) = N_{G'}(C - D)$. If P' is still a bipartite distance-hereditary embedding of P, then P^* is not minimal. In the following we prove that P' is still a bipartite distance-hereditary embedding of P by contradiction again. Assume that P' is not a bipartite distance-hereditary embedding of P, i.e., G' is not a bipartite distance-hereditary graph. There is an induced forbidden subgraph in G'. Let F be the set of vertices that induces an even hole or a domino in G'. Because the induced forbidden subgraph is formed by removing edges connecting a vertex in D and another vertex in $C-D$, $|D \cap F| \geq 1$ and $|(C-D)\cap F| \geq 1$. Since all induced forbidden subgraph are biconnected, $|F \cap N_{G'}(C)| = |F \cap N_{G^*}(C)| \geq 2$. Without loss generality assume that x_1, x_2, x_3, and x_4 are vertices in F where $x_1 \in (C - D)$, $x_2 \in D$, and $x_3, x_4 \in N_{G^*}(C)$. By definition, x_1 and x_2 are not adjacent in G'. By assumption, $x_1, x_2 \in Z_i$ and $x_3, x_4 \in Z_{i-1}$. Clearly $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are fully adjacent both in G^* and in G'. In G' the four vertices x_1, x_2, x_3, and x_4 induce a cycle of length four. Therefore it is impossible for F to induce a hole. Thus F induces a domino. The fifth vertex x_5 and the sixth vertex x_6 of F are adjacent and one of them is adjacent to a vertex in Z_i and the other is adjacent to a vertex in Z_{i-1}. Thus at least one of them is in Z_i or in Z_{i-1}. In G' if it is in Z_i then it is adjacent to both x_3 and x_4 and if it is in Z_{i-1} then it is adjacent to both x_1 and x_2. In other words, F does not induce a domino, a contradiction.

Next we prove Statement (2). By Statement (1), C is also a component of G_i^*. By Theorem 6, $Z_i = L_i$ for $0 \leq i \leq \ell$. Hence $C \cap Z_i \subseteq C \cap L_i$. Since G^* is obtained from G by adding edges, $C \cap L_i \subseteq C \cap Z_i$. Therefore $C \cap Z_i = C \cap L_i$.

Finally, we prove Statement (3). Clearly the statement is true if $i = 1$. In the following assume $1 < i \leq \ell$. By Statement (1) of this lemma, C is also a component of G_i^*. By Statement (2) of this lemma, $C \cap Z_i = C \cap L_i$. Since $|C \cap Z_i| > 1$, by Theorem 6 both $N_{G^*}(C)$ and $C \cap Z_i$ contains probes and nonprobes. Let $x \in C \cap Z_i$ be a probe in G^*. Since G^* is bipartite distance-hereditary, $N_{G^*}(C) = N^-_\Psi(x)$ by Theorem 2. Because $Z_i = L_i$ for $0 \leq i \leq \ell$ (see Theorem 6) and x is a probe in P^*, $N^-_\Psi(x) = N^-_\Phi(x)$. Since G^* is obtained from G by adding edges, $N_G(C) \subseteq N_{G^*}(C)$. Thus $N_{G^*}(C) = N^-_\Psi(x) = N^-_\Phi(x) \subseteq N_G(C) \subseteq N_{G^*}(C)$. This proves the statement. ∎

Theorem 7. *Suppose P^* is a minimal bipartite distance-hereditary embedding of P and C is a component of G_i with $|C \cap L_i| > 1$ and $1 < i < \ell$. A vertex $x \in C \cap L_i$ (resp. $N_G(C)$) is a probe in P^* if and only if x is a adjacent to all vertices in $N_G(C)$ (resp. C).*

Proof: By Statement (2) of Lemma 5, $C \cap Z_i = C \cap L_i$. Hence $|C \cap Z_i| > 1$. By Statement (3) of Lemma 5, $N_G(C) = N_{G^*}(C)$. Since G^* is biconnected, $|N_{G^*}(C)| > 1$. Since G^* is distance hereditary, by Theorem 2 $N_{G^*}(C)$ and $C \cap Z_i$ are fully adjacent. By Lemma 3, the theorem holds. ∎

Next we show how to use the above lemmas and theorems to handle the case that P is of class **C 2**.

Algorithm for D 1. In this case there is a component C in G_2 with $|C \cap L_2| \geq 2$. By Lemma 5 and Theorem 7, a vertex $x \in C \cap L_2$ (resp. $N_G(C)$) is a probe in P^* if and only if x is a adjacent to all vertices in $N_G(C)$ (resp. C). Compute P' from P as follows. Let $P'_G = P_G$ and $P'_L(y) = P_L(y)$ for every $y \in P_V - (C \cup N_G(C))$. For every $y \in (C \cup N_G(C))$, let $P'_L(y) = P_L(y)$ if $P_L(y) \neq \mathbb{U}$. For every $y \in N_G(C)$ with $P_L(y) = \mathbb{U}$, let $P'_L(y) = \mathbb{P}$ if in G y is adjacent to all vertices $z \in C \cap L_2$ and let $P'_L(y) = \mathbb{N}$ otherwise. If we let $P'_L(y) = P_L(y)$ for all primes $y \in C$, we see that P is a probe bipartite distance-hereditary graph if and only if P' is a probe bipartite distance-hereditary graph by Theorem 7. But we will go further. Clearly all vertices in $N_G(C)$ are not primes now. From $i = 2$ to $i = \ell$, for every $y \in C \cap L_i$ with $P_L(y) = \mathbb{U}$, let $P'_L(y) = \mathbb{P}$ if in G y is adjacent to some nonprobes in L_{i-1} and let $P'_L(y) = \mathbb{N}$ otherwise. By Theorem 6 and 7, we see that P

is a probe bipartite distance-hereditary graph if and only if P' is a probe bipartite distance-hereditary graph after we relabel primes of P in C. In P', there must be a probe p' in $C \cap L_2$. Besides $P'_L(y) \neq \mathbb{U}$ for every $y \in N_G(p')$. Thus P' is a well-labeled kernel probe graph. We then call Algorithm **W** to determine whether P' is a probe bipartite distance-hereditary graph. It takes linear time to find a component C of G_2 with $|C \cap L_2| > 1$ and obtain P' in linear time. Thus the algorithm for **D 1** runs in $O(n^2)$ time.

Algorithm for D 2. In this case $\ell = 2$ and there is a vertex $q \in L_2$ with $P_L(q) = \mathbb{P}$. If $N_G(q) = L_1$, then q is a false twin of p, a contradiction. Thus $L_1 - N_G(q) \neq \emptyset$. Let $(L'_0, L'_1, \ldots, L'_k)$ be the hanging of G by q. Then p and all vertices in $L_1 - N_G(q)$ are in $L'_2 + L'_3$ and are in the same component of $G - N_G[q]$. Hence P is also of class **D 1** and the algorithm is finished by calling the algorithm for **D 1**. Thus the algorithm for **D 2** runs in $O(n^2)$ time.

Algorithm for D 3. In this case $\ell = 2$ and every vertex in L_2 is not a probe. Let q be a vertex in L_2 and be of minimum degree among vertices in L_2. By definition, $P_L(q) = \mathbb{U}$ or $P_L(q) = \mathbb{N}$. Let $\hat{P}$ be the probe graph $(P_G, \hat{P}_L)$ where $\hat{P}_L(q) = \mathbb{P}$ and $\hat{P}_L(x) = P_L(x)$ for $x \in P_V - q$. Let $\check{P}$ be the probe graph $(P_G, \check{P}_L)$ where $\check{P}_L(q) = \mathbb{N}$, $\check{P}_L(y) = \mathbb{P}$ for $y \in N_G(q)$, and $\check{P}_L(x) = P_L(x)$ for $x \in P_V - N_G[q]$. If $P_L(q) = \mathbb{U}$, then P is a probe bipartite distance-hereditary graph if and only if one of $\hat{P}$ and $\check{P}$ is a probe bipartite distance-hereditary graph. It is easy to see that we can use the algorithm for **D 2** to test whether $\hat{P}$ is a probe bipartite distance-hereditary graph. In the following we focus on checking whether $\check{P}$ is a probe bipartite distance-hereditary graph. For simplifying the notation, we use G to refer to P_G. Since G has no false twins, $N^-_\Phi(q) \neq N^-_\Phi(q')$ for any $q' \in L_2$ and $q' \neq q$. If for all $q' \in L_2$, $q' \neq q$, either $N^-_\Phi(q) \subset N^-_\Phi(q')$, or $N^-_\Phi(q)$ and $N^-_\Phi(q')$ are disjoint, then any two vertices in $N^-_\Phi(q)$ are false twins in G. Thus there exists a vertex $q' \in L_2$, $q' \neq q$, that $N^-_\Phi(q')$ and $N^-_\Phi(q)$ are incomparable, i.e., $N^-_\Phi(q') \cap N^-_\Phi(q) \neq \emptyset$, $N^-_\Phi(q') - N^-_\Phi(q) \neq \emptyset$, and $N^-_\Phi(q) - N^-_\Phi(q') \neq \emptyset$. Let $y_1, y_2 \in N^-_\Phi(q)$ where $y_2 \in N^-_\Phi(q')$ and $y_1 \notin N^-_\Phi(q')$. Let $z \in N^-_\Phi(q')$ but $z \notin N^-_\Phi(q)$. Notice that y_1 and y_2 must be probes in any bipartite distance-hereditary embedding of $\check{P}$. Consider the hanging $(\check{L}_0, \check{L}_1, \ldots, \check{L}_k)$ of G by y_1. By definition, $d_G(y_1, q') = 3$. Since $d_G(y_1, q') = 3$, $k > 2$. We see that $\check{P}$ is of class **D 1**. Hence whether $\check{P}$ is a probe bipartite distance-hereditary graph can be determined in $O(n^2)$.

The following lemma summarizes the results of this subsection.

Lemma 6. *Whether a probe graph of class* **P 4** *is a probe bipartite distance-hereditary graph can be determined in* $O(n^2)$ *time.*

C. Non-biconnected probe graphs without false twins and Algorithm R

In this subsection we show how to solve the problem recursively when the input probe graph P is bipartite without false twins and is non-biconnected. Our algorithm is based upon the following two theorems.

Theorem 8. [12] *Suppose P is a connected probe graph and P^* is a minimal distance-hereditary embedding of P. Then a vertex is a cut vertex of P^* if and only if it is a cut vertex of P.*

Corollary 7. *Suppose P is a connected probe graph and P^* is a minimal bipartite distance-hereditary embedding of P. Then a vertex is a cut vertex of P^* if and only if it is a cut vertex of P.*

Proof: Suppose P has k biconnected components $C_1, C_2, \ldots, C_k$. Let P^*_G be the graph $(P^*_V, \cup_{j=1}^k P^*_E[C_i]))$. It is easy to see that a vertex is a cut vertex of P^* if and only if it is a cut vertex of P. We then prove the corollary by showing that P^* is indeed a bipartite distance-hereditary embedding of P. If P' is a bipartite distance-hereditary embedding of P and $P' \neq P^*$, then P' is not minimal, a contradiction. Thus $P' = P^*$ if P^* is a bipartite distance-hereditary embedding of P.

Now we prove that P^* is a bipartite distance-hereditary embedding of P. Suppose that P^* is not a bipartite distance-hereditary embedding of P. That is, P^* has a forbidden induced subgraph of bipartite distance-hereditary graphs.

Let F be the vertex set of a forbidden induced subgraph. Since $P^*[C_i]$ is a bipartite distance-hereditary embedding of $P[C_i]$, F is not a subset of any C_i for $1 \leq i \leq k$. Notice that F induces an even hole or a domino. Both forbidden induced subgraphs are biconnected. Thus F must be a subset of some C_i, a contradiction. This completes the proof. ■

Theorem 9. [12] *Let P be a probe graph. If there exists a cut vertex v in P and C is a component of $P_G - v$, then P is a probe distance-hereditary graph if and only if $P - C$ has a distance-hereditary embedding P' and $P[C+v]$ has a distance-hereditary embedding P'' where either $P'_L(v) = P''_L(v) = \mathbb{P}$ or $P'_L(v) = P''_L(v) = \mathbb{N}$.*

Remark 2. *The above theorem also holds for probe bipartite distance-hereditary graphs.*

Theorem 9 points out a recursive way to solve the problem. We now describe Algorithm **R** in detail. Let v be a cut vertex of P_G and C be a component of $P_G - v$ such that C does not contain any other cut vertex of P_G. In other words, $C + v$ induces a biconnected component of P_G. There are two cases:

1. $P_L(v) = \mathbb{P}$ or $\mathbb{N}$. By Theorem 9, P is a probe bipartite distance-hereditary graph if and only if both $P[C + v]$ and $P - C$ are probe bipartite distance-hereditary graphs. Call Algorithm **C** to check whether $P[C + v]$ has an embedding and recursively call the main algorithm to check whether $P - C$ has an embedding.

2. $P_L(v) = \mathbb{U}$. Let $\hat{P}$ be the probe graph $(P_G[C+v], \hat{P}_L)$ where $\hat{P}_L(v) = \mathbb{P}$ and $\hat{P}_L(x) = P_L(x)$ for $x \in C$. Let $\check{P}$ be the probe graph $(P_G[C+v], \check{P}_L)$ where $\check{P}_L(v) = \mathbb{N}$, $\check{P}_L(x) = \mathbb{P}$ for $x \in N_{P_G}(v) \cap C$, and $\check{P}_L(x) = P_L(x)$ for $x \in C - N_{P_G}(v)$. Let P' be the probe graph $(P_G[V-C], P'_L)$ where $P'_L(v) = \mathbb{P}$ and $P'_L(x) = P_L(x)$ for $x \in V - C - v$. Let P'' be the probe graph $(P_G[V-C], P''_L)$ where $P''_L(v) = \mathbb{N}$, $P''_L(x) = \mathbb{P}$ for $x \in N_{P_G}(v) \cap (V - C)$, and $P''_L(x) = P_L(x)$ for $x \in V - C - N_{P_G}[v]$. Call Algorithm **C** to check whether $\hat{P}$ and $\check{P}$ have embeddings. There are four subcases:

(1) If neither $\hat{P}$ nor $\check{P}$ is a probe bipartite distance-hereditary graph, then P is not a probe bipartite distance-hereditary graph.

(2) If both $\hat{P}$ and $\check{P}$ are probe bipartite distance-hereditary graphs, then P is a probe bipartite distance-hereditary graph if and only if $P - C$ is a probe bipartite distance-hereditary graph. Recursively call the main algorithm to check whether $P-C$ has an embedding.

(3) If $\hat{P}$ is a probe bipartite distance-hereditary graph but $\check{P}$ is not, then P is a probe bipartite distance-hereditary graph if and only if P' is a probe bipartite distance-hereditary graph. Recursively call the main algorithm to check whether P' has an embedding.

(4) If $\hat{P}$ is not a probe bipartite distance-hereditary graph but $\check{P}$ is, then P is a probe bipartite distance-hereditary graph if and only if P'' is a probe bipartite distance-hereditary graph. Recursively call the main algorithm to check whether P'' has an embedding.

Definition 4. *A bipartite probe graph P is called a* pseudo-kernel probe graph *if it satisfies one of the following three conditions:*

1) *P is biconnected without false twins.*
2) *P is biconnected and has only one pair of false twins. One of the pair of false twins is not a prime.*
3) *P is non-biconnected without false twins and has only one cut vertex. The cut vertex is not a prime.*

Suppose v is the cut vertex of P_G used to decompose P_G into $P_G[C + v]$ and $P_G - C$ in Algorithm **R**. We use G and $G_C + v$ to denote P_G and $P_G[C + v]$ respectively.

Theorem 10. *Suppose G is a non-biconnected graph without false twins. There exists a biconnected component $G_C + v$ of G that only contains a cut vertex v of G. Then one of the following statements holds.*

(i) *There are no false twins in $G_C + v$.*
(ii) *G_C+v contains exactly one pair of false twins, v is one of the false twins. After removing one of the false twins from $G_C + v$, the resulting graph has no false twins and either it is biconnected or has only one cut vertex which is one of the pair of false twins in $G_C + v$.*

Proof: Since G has no false twins, no $x, y \in G_C$ are false twins in $G_C + v$. After the decomposition, only the neighborhood of the vertex used to decompose the graph is changed. Hence v must be one of the pair of false twins, and there exists only one vertex u in G_C that u and v are false twins in $G_C + v$.

Suppose there exists a pair of false twins x and y in G_C. Removing v from $G_C + v$ only changes the neighborhood of vertices in $N_{G_C}(u)$, where u and v are false twins in G_C+v. Hence one of $x, y \in N_{G_C}(u)$. If $x, y \in N_{G_C}(u)$, they are false twins in $G_C + v$, a contradiction. If $x \in N_{G_C}(u)$ but $y \notin N_{G_C}[u]$, then they are not false twins in G_C since y is not adjacent to u, a contradiction. Similarly, we can show that there are no false twins in $G_C + v - u$.

Suppose G_C is non-biconnected. If u is not a cut vertex, let $x \neq u$ be a cut vertex of G_C. Let C_1 and C_2 be two components of $G_C - x$. Since $G_C + v$ is biconnected, v is adjacent to some vertex of C_1 and some vertex of C_2 in $G_C + v$. Since u and v are false twins in $G_C + v$, in G_C u is adjacent to some vertex of C_1 and some vertex of C_2, a contradiction to the assumption that $x \neq u$ is a cut vertex. Hence u is the only cut vertex in G_C. Similarly, we can show v is the only cut vertex of $G_C + v - u$. ■

Corollary 8. *The probe graph $P[C + v]$ produced in Case 1. of Algorithm* **R** *and the probe graphs $\hat{P}$ and $\check{P}$ produced in Case 2. of Algorithm* **R** *are pseudo-kernel probe graphs. In addition, if u and v are the only pair of false twins in $P_G[C + v]$, after removing a false twin according to Corollary 6 from $P[C+v]$, $\hat{P}$, and $\check{P}$, the resulting probe graph R is a pseudo-kernel probe graph.*

Proof: Note that $P_G[C + v]$ satisfies one of the conditions of Theorem 10 and v is not a prime. Assume u is the false twin of v. By the steps of removing false twins according to Corollary 6, if v is a probe, we remove u; if v is a nonprobe and u is a nonprobe or a prime, we remove u; if v is a nonprobe and u is a probe, we remove v. After removing false twins, if the resulting probe graph R is biconnected, by Theorem 10 it is a kernel probe graph. Assume R is non-biconnected, by Theorem 10 one of u and v is the only cut vertex of R. Moreover, the only cut vertex in R is not a prime. ■

Now we are ready to describe Algorithm **C**. The input of Algorithm **C** is a two-tuple (P, v) where P is a pseudo-kernel probe graph and v is a vertex in P with $P_L(v) = \mathbb{P}$ or $\mathbb{N}$. Note that if P_G is biconnected, it has at most one pair of false twins and v is one of the false twins. If P_G is non-biconnected, v is the only cut vertex in P_G.

Algorithm C. We distinguish the following four classes of the input graphs.

E 1. $|P_V| \leq c$ for some constant c. Solve the problem by brute force in $O(1)$ time.

E 2. P is biconnected without false twins. Call Algorithm **B** to solve the problem in $O(n^2)$ time.

E 3. P_G is biconnected and v has a false twin u. It is easy to see that u can be found in linear time by simply checking whether the open neighborhood of the other vertices is the same as the neighborhood of v. By Corollary 6, we can remove one of u and v from P, and check whether the resulting probe graph is a probe bipartite distance-hereditary graph. We have the following two cases:

- **(1)** $P_L(v) = \mathbb{P}$. Recursively call Algorithm **C** to check $(P - u, v)$.
- **(2)** $P_L(v) = \mathbb{N}$. If $P_L(u) = \mathbb{N}$ or $\mathbb{U}$, recursively call Algorithm **C** to check $(P - u, v)$. If $P_L(u) = \mathbb{P}$, recursively call Algorithm **C** to check $(P - v, u)$.

E 4. P_G is non-biconnected without false twins, v is the only cut vertex in P_G. Let $C_1, C_2, \ldots, C_r$ be biconnected components of P_G. Since v is the only cut vertex in P_G, $C_i \cap C_j = \{v\}$ for $1 \leq i < j \leq r$. For each C_i, $i = 1, 2, \ldots, r$, call Algorithm **C** to check $(P[C_i], v)$.

Lemma 7. *Whether a pseudo-kernel probe graph is a probe bipartite distance-hereditary graph can be checked in $O(nm)$ time.*

Proof: Let $g(n)$ denote the time complexity of Algorithm **C**. We claim that $g(n) \leq c_1 nm$. The input graph of class **E 1** can be recognized in $O(1)$ time. The input graph of class **E 2** can be recognized in $O(n^2)$ time. The input graph of class **E 3** can be recognized in $g(n-1)+c_0(n+m)$ time where $c_0(n+m)$ is the time spent for decomposing the input graph into biconnected components and removing a false twin from it. It is easy to see that $g(n-1)+c_0(n+m) \leq c_1nm$ if $c_1 \geq 2c_0$. The input of class **E 4** can be recognized in $\Sigma_{i=1}^{r} g(n_i) + c_0(n+m)$ time where $n_i = |C_i|$ and $C_1, C_2, \ldots, C_r$ are biconnected components in P_G. Assume that C_r has the maximum number of vertices among $C_1, C_2, \ldots, C_r$.

$$
\begin{aligned}
g(n) &= \Sigma_{i=1}^{r} g(n_i) + c_0(n+m) \\
&\leq \Sigma_{i=1}^{r} c_1 n_i m_i + c_0(n+m) \\
&\leq c_1 n_r \Sigma_{i=1}^{r} m_i + c_0(n+m) \\
&= c_1 n_r m + c_0(n+m) \\
&\leq c_1(n-1)m + c_0(m+m) \\
&\leq c_1 nm - (c_1 - 2c_0)m \\
&\leq c_1 nm \text{ where } c_1 \geq 2c_0.
\end{aligned}
$$

This completes the proof. ■

D. Time complexity

In this subsection we analyze the time complexity of the algorithm.

Theorem 11. *There exists an $O(nm)$-time algorithm to check if a probe graph P is a probe bipartite distance-hereditary graph.*

Proof: By using the data structure described in [19], we can repeat the step of removing false twins until input probe graph P has no false twins in $O(n^2)$ time. If P is biconnected after removing all false twins, then call Algorithm **B** to complete the algorithm. Suppose P is not biconnected after removing all false twins. We go on performing the recursive step that decomposes P into two subgraphs $P_G[C+v]$ and P_G-C (Algorithm **R**). The algorithm calls Algorithm **C** to test $P[C+v]$ (Case 1 of Algorithm **R**) or $\hat{P}$ and $\check{P}$ (Case 2 of Algorithm **R**) and goes on testing the subgraph $P-C$ obtained recursively. Note that removing all false twins from $P-C$ only takes $O(n+m)$ time. After removing C from P only the neighborhood of the cut vertex v is changed. Since P has no false twins, there is only one pair of false twins in $P-C$ and v is one of the false twins. Let $t(n)$ be the time of the whole algorithm. Then $t(n) = t_1(n') + O(n^2)$ where n' is the number of vertices in the input probe graph after removing all false twins and $t_1(n')$ is the time spent by the algorithm after removing all false twins. Let $C_1, C_2, \ldots C_k$ be the biconnected components produced by Algorithm **R** in each recursive call. For each C_i we call Algorithm **C** at most two times. Assume $|C_i| = n_i$ for $i = 1, \ldots, k$. Let m_i denote the number of edges in $P_G[C_i]$. We use $g(n_i)$ to denote the time spent by Algorithm **C** for $i = 1, \ldots, k$.

$$
\begin{aligned}
t_1(n') &= 2g(n_1) + t_1(n' - n_1 + 1) + c_0(n' + m') \\
&= 2g(n_1) + \cdots + 2g(n_k) + c_0 k(n' + m') \\
&= 2\Sigma_{i=1}^{k} g(n_i m_i) + c_0 k(n' + m') \\
&\leq 2\Sigma_{i=1}^{k} c_1 n_i m_i + c_0 k(n' + m') \\
&\leq 2c_1 n' \Sigma_{i=1}^{k} m_i + c_0 k(n' + m') \\
&\leq 2c_1 n' m' + c_0 n'(m' + m') \\
&\leq c_2 n' m' \text{ where } c_2 \geq 4c_1 \\
&= O(n'm')
\end{aligned}
$$

Since $t(n) = t_1(n') + O(n^2)$ and $t_1(n') = O(n'm')$, we have $t(n) = O(nm)$. This completes the proof of the theorem. ■

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