

Generalized Recursive Circulant Graphs*

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Abstract

In this paper, we propose a new class of graphs called *generalized recursive circulant graphs* (GRCGs for short). As it is named, GRCGs are a generalization of recursive circulant graphs. This generalization is achieved by using a multidimensional vertex labeling scheme. Various network metrics of GRCGs, such as degree, connectivity and hamiltonian properties are studied. A shortest path routing algorithm for GRCGs is also proposed in this paper.

Keyword: interconnection network, vertex-symmetric graphs, circulant graphs, recursive circulant graphs, graph algorithm.

1 Introduction

In [7], Boesch and Tindell proposed a class of new graphs called circulant graphs. A *circulant graph* $G(N; s_1, s_2, \dots, s_k)$ has N vertices labeled with integers modulo N where each vertex i is adjacent to $2k$ vertices $i \pm s_1, i \pm s_2, \dots, i \pm s_k$ and $0 < s_1 < s_2 < \dots < s_k < \lfloor N/2 \rfloor$. After that, a wide variety of its related graphs were proposed, e.g., chordal rings [3, 4], recursive circulant graphs [16], etc. A *chordal ring* [4] is a circulant graph $G(N; 1, d)$, or denoted by $CR(N, d)$. By the definition of chordal rings, the degree of every vertex in a chordal ring is 4 except that N is even and $d = N/2$. In this exception, the degree becomes 3. The class of *recursive circulant graphs* (RCGs for

short) was first proposed by Park and Chwa in [16]. An RCG $RC(N, d)$ has $N = cd^m$ vertices, where $0 < c < d$. Every vertex i in $RC(N, d)$ is adjacent to vertices $i \pm d^k \pmod{N}$, where $k = 0, 1, 2, \dots, \lceil \log_d N \rceil - 1$. That is, $RC(N, d) = G(N; d^0, d^1, \dots, d^{\lceil \log_d N \rceil - 1})$ where $N = cd^m$. Since $RC(cd^m, d)$ can be recursively partitioned into d induced subgraphs $RC(cd^{m-1}, d)$, this is the reason why this family of circulant graphs is named as “recursive”. $RC(cd^m, d)$ is regular with degree δ , where δ is $2m - 1, 2m, 2m + 1$ or $2m + 2$, depending on the values of parameters c and d [17]. Figure 1 gives some examples of RCGs with different parameters c and d . Note that $RC(9, 3)$ and $RC(12, 4)$ shown in Figures 1(b) and 1(d), respectively, are both chordal rings while $RC(8, 2)$ and $RC(18, 3)$ are not.

Since RCGs are a subclass of *circulant graphs*, they are vertex-symmetric [9]. As a network topology, RCGs have been widely studied, such as pancyclicity [1], edge-pancyclicity [2], parallel routing algorithm [11], strong hamiltonicity [18], Hamiltonian properties in faulty condition [19], super-connected property [20], and independent spanning trees problem on RCGs [23, 24]. Embeddings of hypercubes and meshes [17], trees [14], full ternary trees [13], and disjoint Hamiltonian cycles [6, 15] were also studied.

A good labeling of the vertices in a graph often results in efficient algorithms for solving the problems in that class of graphs. For example, in hypercubes [5], star graphs [12], generalized hypercubes [5], etc., the label of each vertex is represented as a multidimensional vector. Based on those labeling, a lot of efficient algorithms have been proposed, e.g., routing algo-

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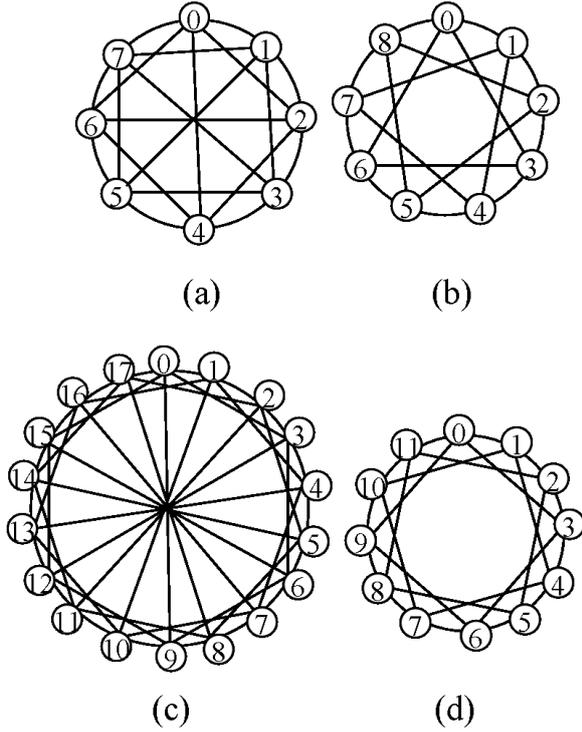


Figure 1: Examples of different recursive circulant graphs: (a) $RC(2^3, 2)$, $c = 1, d = 2, m = 3$; (b) $RC(3^2, 3)$, $c = 1, d = 3, m = 2$; (c) $RC(2 * 3^2, 3)$, $c = 2, d = 3, m = 2$; (d) $RC(3 * 4, 4)$, $c = 3, d = 4, m = 1$.

rithms, Hamiltonicity algorithms, pancyclicity algorithms, etc. Inspired by these labeling systems, we also relabel each vertex of an RCG as a multidimensional vector. For example, Figure 2 shows the two-dimensional vertex labeling of $RC(12, 4)$. Note that the new labels of vertices $0, 1, \dots, 11$ are $(0,0), (0,1), \dots, (2,3)$, respectively.

Besides, we shall extend the definition of RCGs and consequently form a new class of graphs, called *generalized recursive circulant graphs* (GRCGs for short).

The remaining part of this paper is organized as follows. Section 2 presents the definition and basic properties of GRCG. Section 3 presents a routing algorithm for GRCG. The last section contains our concluding remarks.

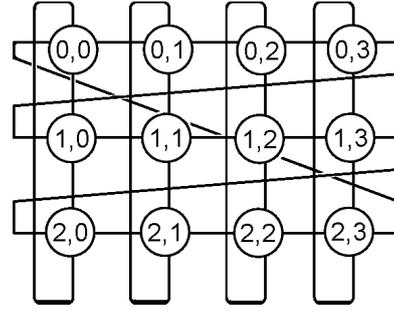


Figure 2: Two dimensional vertex labeling of $RC(12, 4)$.

2 The definition and basic properties of GRCGs

A GRCG, denoted by $GR(m_h, m_{h-1}, \dots, m_2, m_1)$, has $N = \prod_{i=1}^h m_i$ vertices where $m_i \geq 2$ for $1 \leq i \leq h$. An index i is referred to as a *dimension* of the labeling system, while m_i is the *size* of dimension i . Each vertex in the graph is labeled by $(x_h, x_{h-1}, \dots, x_1)$ where $0 \leq x_i \leq m_i$ for $i = 1, 2, \dots, h$. Further, vertex $(x_h, \dots, x_{i+1}, x_i, x_{i-1}, \dots, x_1)$ is adjacent to all of those vertices labeled by $(x_h, \dots, x_{i+1}, x_i \pm 1, x_{i-1}, \dots, x_1)$, where $x_i \pm 1$ is calculated with “carry” and “borrow” when the resulting value is greater than m_i or less than 0, respectively. That is, if $x_i + 1 = m_i$, then x_i becomes 0 and a carry 1 is added to x_{i+1} . If $x_i = 0$, then the operation $x_i - 1$ will borrow 1 from x_{i+1} . The borrow operation results in $x_{i+1} = x_{i+1} - 1$ and $x_i = x_i + m_i$. However, vertex $(x_h, x_{h-1}, \dots, x_{i+1}, x_i - 1, x_{i-1}, \dots, x_1)$ with $x_h = x_{h-1} = \dots = x_i = 0$ is equal to vertex $(m_h - 1, m_{h-1} - 1, \dots, m_{i+1} - 1, m_i - 1, x_{i-1}, \dots, x_1)$. For example, see Figure 3. Figure 3 shows the graph $GR(2, 4, 3)$ with $h = 3$. That is, $m_1 = 3, m_2 = 4$, and $m_3 = 2$. The neighbors of vertex $(1, 3, 0)$ in $GR(2, 4, 3)$ are $(1, 3, 0 + 1), (1, 3, 0 - 1), (1, 3 + 1, 0), (1, 3 - 1, 0), (1 + 1, 3, 0)$, and $(1 - 1, 3, 0)$ which are $(1, 3, 1), (1, 2, 2), (0, 0, 0), (1, 2, 0), (0, 3, 0)$, and $(0, 3, 0)$, respectively.

We recall the vertex adjacency of an RCG. Let v be a vertex in $RC(N, d)$. Then, the neighbors of v are vertices $v \pm d^i \pmod{N}$,

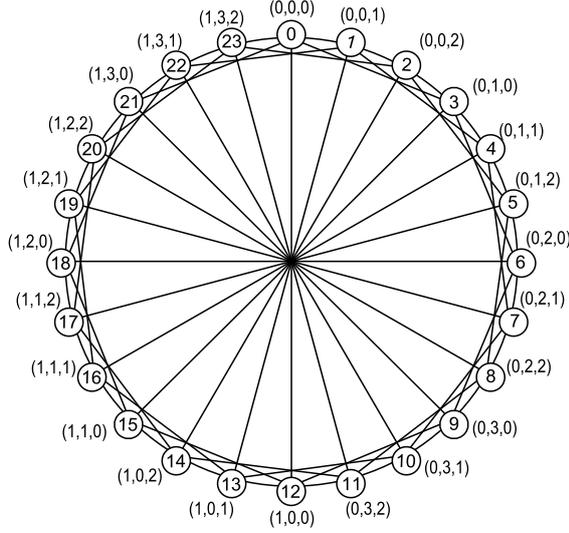


Figure 3: A generalized recursive circulant graph $GR(2, 4, 3)$

where $0 \leq i \leq \lceil \log_d N \rceil - 1$. The increment $+d^i$ or the decrement $-d^i$ is called a *jump* of v . We adapt the meaning of jump to GRCGs. Let j_i be a *jump* of dimension i . Then, for any vertex in $GR(m_h, \dots, m_2, m_1)$, jump $j_1 = 1$ and $j_i = \prod_{k=1}^{i-1} m_k$ for $i = 2, 3, \dots, h$.

Let $x = (x_h, x_{h-1}, \dots, x_1)$ be a vertex in $GR(m_h, m_{h-1}, \dots, m_1)$. The *serial number* of x , denoted by $s(x)$, is a unique number between 0 to $N - 1$ that represents the original decimal label of x . Based on the definition of GRCGs, $s(x)$ is obtained by using the formula:

$$s(x) = x_1 + \sum_{i=2}^h (x_i \cdot j_i).$$

Conversely, a serial number number $s(x)$ between 0 to $N - 1$ can be transformed to a h -dimensional label $(x_h, x_{h-1}, \dots, x_1)$ by setting

$$x_i = \left\lfloor \frac{s(x) - \sum_{k=i+1}^h x_k \cdot j_k}{j_i} \right\rfloor,$$

for $i = h, h - 1, \dots, 1$.

For example, the serial number of $x =$

$(1, 3, 0)$ in $GR(2, 4, 3)$ is:

$$\begin{aligned} s(x) &= x_1 + x_2 \cdot j_2 + x_3 \cdot j_3 \\ &= x_1 + x_2 \cdot m_1 + x_3 \cdot m_2 \cdot m_1 \\ &= 0 + 3 \cdot 3 + 1 \cdot 4 \cdot 3 \\ &= 0 + 9 + 12 \\ &= 21. \end{aligned}$$

Conversely, given $s(x) = 21$, we can compute $x_3 = \lfloor s(x)/j_3 \rfloor = \lfloor 21/12 \rfloor = 1$, $x_2 = \lfloor (s(x) - x_3 \cdot j_3)/j_2 \rfloor = \lfloor (21 - 1 \cdot 12)/3 \rfloor = \lfloor (21 - 12)/3 \rfloor = 3$, and $x_1 = \lfloor (s(x) - \sum_{k=2}^3 (x_k \cdot j_k))/j_1 \rfloor = \lfloor (21 - 12 - 9)/1 \rfloor = 0$.

From above discussion, we can figure out that an RCG $RC(cd^m, d)$ is isomorphic to an $(m + 1)$ -dimensional GRCG $GR(c, d, d, \dots, d)$ (in case that $c > 1$) or an m -dimensional GRCG $GR(d, d, \dots, d)$ (in case that $c = 1$). Therefore, RCG is a subclass of GRCG. Since each vertex in a GRCG has the same jump set, GRCG form a subclass of circulant graphs.

Moreover, we review the recursive structure of RCG. Let V_i be a vertex set in $RC(cd^m, d)$ such that $V_i = \{v \mid v \equiv i \pmod{d}\}$. For $0 \leq i \leq d - 1$, the subgraph induced by V_i is isomorphic to $RC(cd^{m-1}, d)$. For example, $RC(18, 3)$ shown in Figure 1(c) contains three disjoint copies of $RC(6, 3)$. Further, any induced subgraph contains exactly those vertices congruent to modulo 3. In addition, the *basic cycle* of an RCG is the cycle that consists of all those edges not in the induced subgraphs [6]. The basic cycle of an RCG contains precisely the edges of the form $(i, i + 1 \pmod{N})$ and forms a Hamiltonian cycle in the graph.

Suppose the recursive structure still holds in GRCGs. A GRCG can be decomposed recursively. That is, $GR(m_h, \dots, m_2, m_1)$ can be partitioned into m_1 subgraphs isomorphic to $GR(m_h, \dots, m_2)$. The partition process can be continued, and for all $i \leq h$, $GR(m_h, \dots, m_i)$ is still a GRCG. Finally, $GR(m_h)$ is a cycle with m_h number of vertices (if $m_h > 2$) or a 2-clique (if $m_h = 2$).

We denote by δ_h the degree of $GR(m_h, \dots, m_2, m_1)$. Then, it is obvious that $\delta_h = \delta_{h-1} + 2$ when $h \geq 2$. The close

form of δ_h is shown as follows:

$$\delta_h = \begin{cases} 2h, & \text{if } m_h > 2; \\ 2h - 1, & \text{if } m_h = 2. \end{cases}$$

The connectivity and edge connectivity of a GRCG are the same as its degree. Since GRCGs form a subclass of circulant graphs, they are vertex-symmetric. But most of them are not edge-symmetric.

A graph G is *hamiltonian connected* if every two vertices of G are connected by a hamiltonian path [8]. Necessarily, a Hamiltonian connected graph cannot be a bipartite. A bipartite graph G with equal partite sets is *hamiltonian-laceable* if there is a Hamiltonian path between every pair of vertices that separately belong to different partite sets of G [21].

As for the hamiltonian property of GRCGs, we apply an important theorem proposed by Chen et al. in [10] and obtain the following lemma.

Lemma 2.1. *A GRCG is hamiltonian connected if it is neither a bipartite graph nor a cycle.*

Proof. By Chen's theorem [10], a Cayley graph is hamiltonian connected if it is not a bipartite graph, nor a cycle. Since circulant graphs are Cayley graphs, the lemma holds. \square

We have known that $GR(m_1)$ is a cycle. The following lemma provides a simple rule to identify bipartite GRCGs.

Lemma 2.2. *$GR(m_h, m_{h-1}, \dots, m_1)$ with $h \geq 2$ is bipartite if and only if m_h is even and m_{h-1}, \dots, m_1 are odd.*

Proof. Suppose a vertex v in the graph belongs to different vertex sets according to whether $s(v)$ is even or odd. For the sufficiency, if m_h is even, as well as m_{h-1}, \dots, m_1 are odd, jumps $\pm 1, \pm m_1, \pm m_2 m_1, \dots$ and $\pm \prod_{i=1}^{h-1} m_i$ connect a pair of vertices that separately belong to different vertex sets. Thus, $GR(m_h, m_{h-1}, \dots, m_1)$ is a bipartite graph.

For the necessity, we should know that there is no induced odd cycle in a bipartite graph.

The basic cycle of $GR(m_h, m_{h-1}, \dots, m_1)$ is even. That is, $\prod_{i=1}^h m_i$ must be even. Since a jump $+$ $\prod_{i=1}^p m_i$ from any vertex v followed by $\prod_{i=1}^p m_i$ number of jumps -1 also forms a cycle with $\prod_{i=1}^p m_i + 1$ vertices. That is, $\prod_{i=1}^p m_i$ must be odd for $1 \leq p \leq h - 1$. Therefore, m_h must even and m_{h-1}, \dots, m_1 must be odd. \square

An $r \times c$ rectangular grid G is bipartite. We call a vertex in G a *corner vertex* if its degree is two. In [10], the authors proved that if rc is even, then G has a hamiltonian path from any corner vertex to any other vertex in different partite set. We employ this result to prove the following lemma.

Lemma 2.3. *A bipartite GRCG $GR(m_h, m_{h-1}, \dots, m_1)$ with $h \geq 2$ is hamiltonian-laceable.*

Proof. Let $N = \prod_{i=1}^h m_i$ be the number of vertices in $GR(m_h, m_{h-1}, \dots, m_1)$. Let $r = m_h$ and $c = \prod_{i=1}^{h-1} m_i$. According to Lemma 2.4, $N = rc$ is even. Since GRCGs are vertex-symmetric, without loss of generality, we assign vertex $(0, \dots, 0)$ (all dimensions are 0, hereafter called "vertex 0") as a corner vertex, and embed an $r \times c$ grid into the GRCG. The embedding is achieved by taking two types of jumps from vertex 0 to every other vertices. That is, jumps $+1$ (or j_1) can form a horizontal path from vertex 0 to vertex $(0, m_{h-1} - 1, \dots, m_1 - 1)$. Meanwhile, jumps $+$ $\prod_{i=1}^{h-1} m_i$ (or j_h) can form a vertical path from vertex 0 to vertex $(m_h - 1, 0, \dots, 0)$. Then, the product graph of the two paths is mapped to the grid (guest graph). By the result proposed in [10], there exists a hamiltonian path from vertex 0 to any other vertex that belongs to a different partite set. This completes the proof. \square

By combining Lemmas 2.1, 2.2 and 2.3, we have the following theorem.

Theorem 1. *$GR(m_h, m_{h-1}, \dots, m_1)$ with $h \geq 2$ is either hamiltonian connected or hamiltonian-laceable.*

3 Routing algorithm in GR-CGs

In this section, we develop a shortest-path routing algorithm in a GRCG. Let $x = (x_h, x_{h-1}, \dots, x_1)$ be a vertex in $GR(m_h, m_{h-1}, \dots, m_1)$ with $h \geq 2$. We design an algorithm that can find a shortest path from x to vertex 0. Since GRCGs are vertex-symmetric, without loss of generality, we simply consider vertex 0 as the end vertex of a shortest path, and denote a shortest path of x by P_x . For simplicity, P_x is represented by a set of jumps (not necessarily distinct). We denote by J_x the set of distinct jumps in P_x . For $j \in J_x$, we denote by $n(j)$ the number of occurrences of j appeared in P_x . Let $\dim(j)$ denote the dimension of jump j . It is quite obvious that the following properties hold:

- (i) one vertex cannot be visited twice in a shortest path,
- (ii) j and $-j$ cannot coexist in J_x , and
- (iii) $n(j) \leq \lfloor \frac{m_{\dim(j)}}{2} \rfloor$ for every $j \in J_x$.

The basic idea on finding P_x is to reduce all dimension values to 0 with the minimum number of jumps. In case of $m_p \geq 3$ ($1 \leq p \leq h$), the decision is easy to make. That is, if $x_p \leq \lfloor \frac{m_p}{2} \rfloor$, a negative jump in dimension p ($-j_p$) is taken until x_p is reduced to 0. Otherwise, a positive jump j_p is taken. Variable *Carry* is set to 1 when a positive jump increases x_p to m_p (congruent to 0).

We can perform the following procedure to generate the jump set J_x and counts $n(j)$ for jump $j \in J_x$.

Procedure SHORTEST-PATH(x)

begin

1. $J_x = \emptyset$; $Carry = 0$;
2. **For** $i = 1$ to h **do**
3. **if** $0 < x_i \leq \lfloor m_i/2 \rfloor$
4. $J_x = J_x \cup \{-j_i\}$; $n(-j_i) = x_i$;
5. $Carry = 0$;
6. **else if** $\lfloor m_i/2 \rfloor < x_i \leq m_i - 1$
7. $J_x = J_x \cup \{j_i\}$; $n(j_i) = m_i - x_i$;
8. $x_{i+1} = x_{i+1} + 1$;
9. $Carry = 1$;

10. **else**
 11. **if** $Carry = 1$
 12. $x_{i+1} = x_{i+1} + 1$;
 13. **endif**
 14. **endif**
 15. **enddo**
- end SHORTEST-PATH**

Note that the addition operations in lines 8 and 12 of Procedure SHORTEST-PATH are both taken modulo m_{i+1} .

Procedure SHORTEST-PATH can be applied to the case of $m_p = 2$, but it requires that $m_{p+1} \geq 3$ and $m_{p-1} \geq 3$ if they exist. For example, a shortest path of $x = (0, 2, 1)$ in $GR(2, 4, 3)$ (See Figure 3) goes through $(0, 2, 0)$ and $(0, 1, 0)$. That is, $P_x = \{-j_1, -j_2, -j_2\}$, $J_x = \{-j_1, -j_2\}$, while $n(-j_1) = 1$ and $n(-j_2) = 2$.

In case of consecutive dimensions with $m_i = 2$ ($p \geq i \geq q$, where $h \geq p, q \geq 1$, and $p > q$), the decision selection of jumps becomes more complex. Since the partial vector $(x_p, x_{p-1}, \dots, x_{q+1}, x_q)$ is a binary representation, it can be separately viewed as a vertex in $RC(2^{p-q+1}, 2)$. The routing algorithm for a vertex in $RC(2^m, 2)$ can refer to [24]. Briefly, two candidate paths (called *elementary paths*) are constructed in this case. Finally, the path with less jumps is chosen as the solution P_x .

4 Concluding remarks

In this paper, we propose a new class of circulant graphs, the GRCGs, as a generalization of the well-known RCGs. With some properties similar to multidimensional tori, the topology of GRCGs is suitable for the design of parallel computers. Some network metrics of GRCGs, such as diameter, mean internode distance, node visit ratio, etc., ought to be studied later.

Based on the structure characterizations, GRCGs present more flexibility than RCGs in adjusting the number of vertices. Further, many algorithms developed for RCGs, such as independent spanning trees, super-connected

property, pancyclicity, and hamiltonian decomposition, and so forth, can be adapted to GR-CGs.

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