The Study for Adjacent Vertices Fault-Tolerance Bifanability of Hypercube

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***Abstract*—Let $Q_n = (V_b \cup V_w, E)$ be the n-dimensional hypercube. An Let F_a be the set of f_a pairs of adjacently faulty vertices. Let $s_1, t_2^1, t_1^2, \cdots, t_1^{k_1} \in V_b, s_2, t_1^1, t_2^2, \cdots, t_2^{k_2} \in V_w$ be arbitrary fault-free vertices of Q_n. In this paper, we construct the spanning internally disjoint paths $P(s_1, t_1^i)$ and $P(s_2, t_2^j)$ of $Q_n - F_a$ for $f_a + k_1 + k_2 \le n-1$ and $1 \le i \le k_1, 1 \le j \le k_2$.**

***Index Terms*—hypercube, vertices fault-tolerance, fanability, bifanability, Hamiltonian-laceable**

I. Introduction

The hypercube network is one of the most popular interconnection networks. It has many attractive properties, such as regularity, symmetry, small degree and diameter, maximum fault tolerance, easy routing algorithms.

Network topology is usually represented by a graph where vertices represent processors and edges represent links between processors. A *bipartite* graph $G = (V_b \cup V_w, E)$ is a graph such that $V_b \cap V_w = \emptyset$ and every $e = (u, v) \in E, u \in V_b$ and $v \in V_w$. Let $V_b(V_w)$ be the set of black(white) vertices. An n-dimensional hypercube Q_n is a bipartite graph with 2^n vertices. Each vertex is denoted as an n-bit binary string. Two vertices of hypercube are adjacent if and only if their binary string representations differ exactly in one bit position. A *path* is a sequence of adjacent vertices, denoted as $\langle v_1, v_2, \cdots, v_n \rangle$, where all the vertices except v_1 and v_n are distinct. A *cycle*, written as $\langle v_1, v_2, ..., v_n \rangle$, is a path for $v_1 = v_n$. A *Hamiltonian cycle*(*path*) is a cycle(path) that visits every vertex exactly once. A graph G is *Hamiltonian* if G contains a Hamiltonian cycle. A bipartite graph $G = (V_b \cup V_w, E)$ is *Hamiltonian laceable* if there exists a Hamiltonian path between each pair of vertices u and v for $u \in V_b$ and $v \in V_w$. A bipartite graph $G = (V_b \cup V_w, E)$ is *hyper-Hamiltonian laceable* if there exists a Hamiltonian path between each pair of vertices u and v of $G - \{y\}$ for $u, v \in V_i, z \in V_j, \{i, j\} = \{b, w\}$. A graph G is k *edges hyper-Hamiltonian laceable* if $G - F_e$ is hyper-Hamiltonian laceable $\forall F_e \subset E$ and $|F_e| = k$. A graph G is *k edges hyper-Hamiltonian laceable* if $G - F_e$ is hyper-Hamiltonian laceable. $\forall F_e \subset E$ and $|F_e| = k$.

The following concepts are introduced in [2]. A k-*container* $C(s, t)$ of a graph G is a set of k internal vertex disjoint paths between s and t. Let $V(C(s, t))$ be the set of vertices incident with some paths in $C(s, t)$. A k^*-*container* $C(s, t)$ is a k-container that $V(C(s, t)) = V(G)$. A k^*-*laceable* graph is a bipartite graph that there exists a k^*-container between every two vertices with different color. Thus, a graph is 1^*-laceable and 2^*-laceable if and only if it is Hamiltonian laceable. In [2], Chang et al. showed that Q_n is k^*-laceable for $1 \le k \le n$.

In [3], Chen et al. proposed some more general concepts. A k-fan $A(s \to T)$ of a bipartite graph $G = (V_b \cup V_w, E)$ is a set of k paths from $s \in V_b$ and $T = \{t_i | t_1 \in V_w$, and $t_j \in V_b$, for $2 \le j \le k\}$, such that every two paths of them share only the vertex s. Let $V(A(s \to T))$ be the set of vertices incident with some paths in $A(s \to T)$. A k^*-fan (s, T) of G is a k-fan with $V(A(s \to T)) = V(G)$. A graph G is k^*-*fanable*, if $\forall s \in V_i$ and $\forall T = \{t_1, t_2, \cdots, t_k | t_1 \in V_j$ and $t_2, t_3, \cdots, t_k \in V_i\}$ for $\{i, j\} = \{b, w\}$, there exists a k^*-fan. A graph G is f_e *edges* k^*-*fanable* if $G - F_e$ is k^*-fanable $\forall F_e \subset E(G)$ with $|F_e| = f_e$. Chen et al. proved that Q_n is f_e edges k^*-fanable for $n \geq 3, 0 \leq f_e \leq n-2, 1 \leq k+f_e \leq n$. Let F_a be the set of f_a pairs of adjacently faulty vertices. In this paper, we will prove that $Q_n - F_a$ is (n-f_a)*-fanable.

In this paper, we furthermore investigate the fanability with two source vertices, named bifanability. A k-bifan $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of a bipartite graph $G = (V_b \cup V_w, E)$ is a set of k paths constructed by $|T_1|$-fan $A(s_1 \rightarrow T_1)$ and $|T_2|$-fan $A(s_2 \rightarrow T_2)$ for $|T_1| + |T_2| = k$. Let $V(B(s_1 \rightarrow T_1, s_2 \rightarrow T_2))$ be the set of vertices incident with some paths in $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$. A k^*-bifan $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of G is a k-fan with $V(B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)) = V(G)$. A graph G is ***k^*-bifanable***, if $\forall s_1 \in V_x, s_2 \in V_y$ and $\forall T_1 = \{t_1^1, t_1^2, \cdots, t_1^i | t_1^1 \in V_y$ and $t_1^2, t_1^3, \cdots, t_1^i \in V_x\}$ and $T_2 = \{t_2^1, t_2^2, \cdots, t_2^j | t_2^1 \in V_x$ and $t_2^2, t_2^3, \cdots, t_2^j \in V_y\}$ for $\{x, y\} = \{b, w\}$ and $i + j = k$, there exists a k^*-bifan. In this paper, we will prove that $Q_n - F_a$ is (n-1-f_a)*-bifanable of hypercube.

II. The Adjacent Vertices Fault-Tolerance of Fanability

In this section, we will prove the adjacent vertices fault-tolerance of fanability of hypercube.

We first need to define some notations. Let F_a be the set of f_a pairs of adjacently faulty vertices. Let V^i be the vertex set of Q^i_{n-1}, for $i = 0, 1$. Let $F^i_a = F_a \cap V^i$ and f^i_a be the number of pairs of adjacently vertices of F^i_a for $i = 0, 1$.

The following lemma is introduced in [1].

Lemma 1: The graph Q_n is f-adjacency ($n-2-f$) edges Hamiltonian for $0 \leq f \leq (n-2)$, f-adjacency ($n-2-f$) edges Hamiltonian laceable for $0 \leq f \leq (n-3)$, and f-adjacency ($n-3-f$) edges hyper-Hamiltonian laceable for $0 \leq f \leq (n-3)$.

The following lemma is proposed in [3].

Lemma 2: The Q_n is f_e edges k^*-fanable for $n \geq 3, 0 \leq f_e \leq n-2$ and $1 \leq k \leq n - f_e$.

The following lemma is proved in [5].

Lemma 3: The graph Q_n is f-adjacency $(n-3-f)$ edges property 2H, for $0 \leq f \leq n-3, n \geq 3$.

Theorem 1: The graph Q_n is f_a-adjacency $(n - f_a)^*$-fanable, for $0 \leq f_a \leq n-3$, $n \geq 3$.

Proof: We will prove this theorem by induction on n. Applying Lemma 2, we can obtain that this theorem is true for $f_a = 0$. The graph Q_4 can be verified to be 1-adjacency 3^*-fanable by brute force. Thus, this theorem is true for $n \leq 4$. Without loss of generality, we can assume that $s \in V^0$. In the follows, we will assume that $f_a \geq 1$ and $n \geq 5$.

Case 1 $t_1 \in V^0$.

Case 1.1 $|T^1| = 0$ and $f^1_a = 0$.

Case 1.1.1 $f^0_a \leq n-4$.

Since Q^0_{n-1} is f_a-adjacency $(k-1)^*$-fanable, there exists a $(k-1)^*$-fan $A(s \rightarrow \{t_1, \cdots, t_{k-1}\})$ of $Q^0_{n-1} - F^0_a$. Thus, t_k is on some path of $A(s \rightarrow \{t_1, \cdots, t_{k-1}\})$. Without loss of generality, we can assume t_k is on $P(s, t_{k-1})$. Thus, the path $P(s, t_{k-1})$ can be written as $\langle s \stackrel{P(s,t_k)}{\longrightarrow} t_k, a \stackrel{P(a,t_{k-1})}{\longrightarrow} t_{k-1} \rangle$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(s), \phi(a))$ of Q^1_{n-1}. Hence, $A(s \rightarrow \{t_1, \cdots, t_{k-1}\}) \cup \{\langle s, \phi(s) \stackrel{P(\phi(s),\phi(a))}{\longrightarrow} \phi(a), a \stackrel{P(a,t_{k-1})}{\longrightarrow} t_{k-1} \rangle, P(s, t_k)\} - \{P(s, t_{k-1})\}$ is a k^*-fan of $Q_n - F_a$, as illustrated in Figure 1 (a).

Case 1.1.2 $f^0_a = n-3$.

Applying Lemma 1, we can construct a Hamiltonian cycle in $Q^0_{n-1} - F_a$. Without loss of generality, we can assume that this cycle is denoted as $\langle t_1, a_1 \stackrel{P(a_1,t_2)}{\longrightarrow} t_2, a_2 \stackrel{P(a_2,t_3)}{\longrightarrow} t_3, a_3 \stackrel{P(a_3,s)}{\longrightarrow} s \stackrel{P(s,t_1)}{\longrightarrow} t_1 \rangle$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s), \phi(a_2))$ and $P(\phi(a_3), \phi(a_1))$. Thus, the three disjoint paths $\langle s \stackrel{P(s,a_3)}{\longrightarrow} a_3, \phi(a_3) \stackrel{P(\phi(a_3),\phi(a_1))}{\longrightarrow} \phi(a_1), a_1 \stackrel{P(a_1,t_2)}{\longrightarrow} t_2 \rangle, P(s, t_1)$ and $\langle s, \phi(s) \stackrel{P(\phi(s),\phi(a_2))}{\longrightarrow} \phi(a_2), a_2 \stackrel{P(a_2,t_3)}{\longrightarrow} t_3, \rangle$ form the 3^*-fan $A(s \rightarrow \{t_1, t_2, t_3\})$ of $Q_n - F_a$, as illustrated in Figure 1 (b).

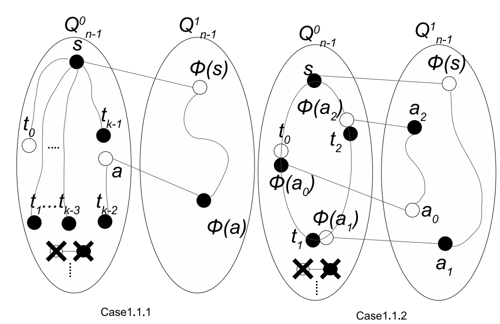

Fig. 1. Illustration of **Case 1.1**

Case 1.2 $|T^1| = 0$ and $1 \leq f^1_a \leq n-4$.

Since $f_a^1 \geq 1$, $Q_{n-1}^0 - F_a^0$ is k^*-fanable. There exists a k^*-fan $A^0(s \rightarrow T)$ of $Q_{n-1}^0 - F_a^0$. Without loss of generality, we can assume that the path $P(s, t_k) = \langle s \xrightarrow{P(s,a)} a, b \xrightarrow{P(b,t_k)} t_k \rangle$ for $\phi(a), \phi(b) \notin F_a^1$. Since $f_a^1 \leq n-4$, $Q_{n-1}^1 - F_a^1$ is Hamiltonian laceable. There exists a Hamiltonian path $P(\phi(a), \phi(b))$ of $Q_{n-1}^1 - F_a^1$. Thus, $A^0(s \rightarrow T) - \{P(s, t_k)\} \cup \{\langle s \xrightarrow{P(s,a)} a, \phi(a) \xrightarrow{P(\phi(a),\phi(b))} \phi(b), b \xrightarrow{P(b,t_k)} t_k \rangle\}$ is a k^*-fan of $Q_n - F_a$, as illustrated in Figure 2 (a).

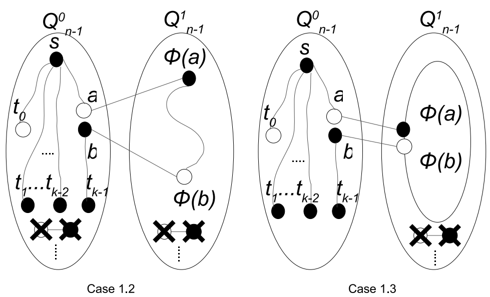

Fig. 2. Illustration of **Case 1.2 and Case 1.3**

Case 1.3 $|T^1| = 0$ and $f_a^1 = n-3$.
Applying Lemma 1, we can construct a Hamiltonian cycle C of $Q_{n-1}^1 - F_a^1$. We can choose a neighbor of t_3, denoted a, such that the cycle $C = \langle \phi(a), \phi(b) \xrightarrow{P(\phi(b),\phi(a))} \phi(a) \rangle$ for $b \notin \{t_2, t_3\}$. By induction hypothesis, there exists a 3*-fan $A(s \rightarrow \{t_1, t_2, b\})$ of $Q_{n-1}^0 - \{t_3, a\}$. Thus, $A(s \rightarrow \{t_1, t_2, b\}) - \{P(s, b)\} \cup \{\langle s \xrightarrow{P(s,b)} b, \phi(b) \xrightarrow{P(\phi(b),\phi(a))} \phi(a), a, t_3 \rangle\}$ is a 3*-fan of $Q_n - F_a$, as illustrated in Figure 2 (b).

Case 1.4 $f_a^0 = n-3$ and $|T^1| = 1$.
Without loss of generality, we can assume that $t_3 \in V^1$. Applying Lemma 1, we can obtain a Hamiltonian cycle C of $Q_{n-1}^0 - F_a^0$. Suppose that (t_1, t_2) is an edge of C. Thus, there exist two spanning disjoint paths $P(s, t_1)$ and $P(s, t_2)$ of $Q_{n-1}^0 - F_a^0$. There also exists a Hamiltonian path $P(\phi(s), t_3)$ of Q_{n-1}^1. These three paths form the 3*-fan $A(s \rightarrow T)$ of $Q_n - F_a$. Suppose that (t_1, t_2) is not an edge of C. We can denote C as $\langle s \xrightarrow{P(s,t_1)} t_1, a \xrightarrow{P(a,b)} b, t_2 \xrightarrow{P(t_2,s)} s \rangle$. Suppose that $\phi(b) \neq t_3$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s), \phi(b))$ and $P(\phi(a), t_3)$ of Q_{n-1}^1. Thus, $P(s, t_1), P(t_2, s)$ and $\langle s, \phi(s) \xrightarrow{P(\phi(s),\phi(b))} \phi(b), b \xrightarrow{P(b,a)} a, \phi(a) \xrightarrow{P(\phi(a),t_3)} t_3 \rangle$ form a 3*-fan $A(s \rightarrow T)$ of $Q_n - F_a$. Suppose that $\phi(b) = t_3$. Applying Lemma 1, we can obtain a Hamiltonian path $P(\phi(s), \phi(a))$ of $Q_{n-1}^1 - \{t_3\}$. Thus, $P(s, t_1), P(t_2, s)$ and $\langle s, \phi(s) \xrightarrow{P(\phi(s),\phi(a))} \phi(a), a \xrightarrow{P(a,b)} b, t_3 \rangle$ form a 3*-fan $A(s \rightarrow T)$ of $Q_n - F_a$.

Case 1.5 $f_a^0 = n-3$ and $|T^1| = 2$.
Applying Lemma 1, we can construct a Hamiltonian cycle $\langle s \xrightarrow{P(s,t_1)} t_1, a \xrightarrow{P(a,s)} s \rangle$ of $Q_{n-1}^0 - F_a^0$. Applying Lemma 3, we can obtain two spanning disjoint paths $P(\phi(s), t_2)$ and $P(\phi(a), t_3)$ of Q_{n-1}^1. Thus, $P(s, t_1), \langle s, \phi(s) \xrightarrow{P(\phi(s),t_2)} t_2 \rangle$ and $\langle s \xrightarrow{P(s,a)} a, \phi(a) \xrightarrow{P(\phi(a),t_3)} t_3 \rangle$ form a 3*-fan $A(s \rightarrow T)$ of $Q_n - F_a$.

Case 1.6 $f_a^0 \leq n-4$ and $|T^1| = 1$.
Without loss of generality, we can assume that $t_k \in V^1$. By induction hypothesis, Q_{n-1}^0 is f_a^0-adjacency $(n-1-f_a^0)$-bifanable for $f_a^0 \leq n-4$. There exists a $(k-1)^*$-fan $A(s \rightarrow \{t_1, \cdots, t_{k-1}\})$ of $Q_{n-1}^0 - F_a^0$. Applying Lemma 1, we can obtain a Hamiltonian path $P(\phi(s), t_k)$ of Q_{n-1}^1. Thus, $A(s \rightarrow \{t_1, \cdots, t_{k-1}\}) \cup \{\langle s, \phi(s) \xrightarrow{P(\phi(s),t_k)} t_k \rangle\}$ is a k^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$.

Case 1.7 $f_a^0 \leq n-4$ and $|T^1| \geq 2$.
Without loss of generality, we can assume that $t_{j+1}, \cdots, t_k \in V^1$. Let $(\phi(a_i), t_i)$ be edges of Q_{n-1}^1 such that $a_i \notin (F_a^0 \cup \{t_2, \cdots, t_j\})$ for $j+1 \leq i \leq k-2$. Let $\phi(b_{k-1})$ be a white node of Q_{n-1}^1 for $b_{k-1} \notin F_a^0$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s), t_k)$ and $P(\phi(b_{k-1}), t_{k-1})$ of $Q_{n-1}^1 - \{t_m, \phi(a_m)|$ for $2 \leq m \leq k-2\}$. By induction hypothesis, there exists a $(k-1)^*$-fan $A(s \rightarrow \{t_1, t_2, \cdots, t_j, a_{j+1}, \cdots, a_{k-2}, b_{k-1}\})$ of $Q_{n-1}^0 - F_a^0$. Thus, the following paths $\langle s, \phi(s) \xrightarrow{P(\phi(s),t_k)} t_k \rangle, \langle s \xrightarrow{P(s,b_{k-1})} b_{k-1}, \phi(b_{k-1}) \xrightarrow{P(\phi(b_{k-1}),t_{k-1})} t_{k-1} \rangle$, $P(s, t_m)$ and $\langle s \xrightarrow{P(s,a_i)} a_i, \phi(a_i), t_i \rangle$ form a k^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$ for $1 \leq m \leq j$ and $j+1 \leq i \leq k-2$.

Case 1.8 $|T^1| = 1$ and $f_a^1 \leq n-4$.
Without loss of generality, we can assume that

$t_k \in V^1$. By induction hypothesis, there exists a k^*-fan $A(s \to \{t_1, \cdots, t_{k-1}, a_k\})$ of $Q^0_{n-1} - F^0_a$ for $a_k, \phi(a_k) \notin F_a$. Applying Lemma 1, we can obtain a Hamiltonian path $P(\phi(a_k), t_k)$ of $Q^1_{n-1} - F^1_a$. Thus, $A(s \to \{t_1, \cdots, t_{k-1}, a_k\}) - \{P(s, a_k)\} \cup \{\langle s \xrightarrow{P(s,a_k)} a_k, \phi(a_k) \xrightarrow{P(\phi(a_k),t_k)} t_k\rangle\}$ is a k^*-fan $A(s \to T)$ of $Q_n - F_a$.

Case 1.9 $|T^1| \geq 2$ and $f^1_a + |T^1| \leq n - 2$.
Without loss of generality, we can assume that $t_{j+1}, \cdots, t_k \in V^1$. Let $(\phi(a_i), t_i)$ be edges of Q^1_{n-1} such that $a_i \notin (F^0_a \cup \{t_2, \cdots, t_j\})$ for $j+1 \leq i \leq k$. By induction hypothesis, there exists a k^*-fan $A(s \to \{t_1, t_2, \cdots, t_j, a_{j+1}, \cdots, a_k\})$ of $Q^0_{n-1} - F^0_a$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(a_k), t_k)$ and $P(\phi(a_{k-1}, t_{k-1})$ of $Q^1_{n-1} - \{t_m, \phi(a_m)|$ for $j+1 \leq m \leq k-2\}$. Thus, the following paths $P(s, t_m), \langle s \xrightarrow{P(s,a_i)} a_i, \phi(a_i), t_i\rangle$ and $\langle s \xrightarrow{P(s,a_r)} a_r, \phi(a_r) \xrightarrow{P(\phi(a_r),t_r)} t_r\rangle$ form a k^*-fan $A(s \to T)$ of $Q_n - F_a$ for $1 \leq m \leq j, r = k-1, k$ and $j+1 \leq i \leq k-2$.

Case 1.10 $|T^1| = 1$ and $f^1_a = n - 3$.
Without loss of generality, we can assume that $t_3 \in V^1$. Applying Lemma 1, we can obtain a Hamiltonian cycle $\langle t_3, \phi(a_3) \xrightarrow{P(\phi(a_3),t_3)} t_3\rangle$ of $Q^1_{n-1} - F^1_a$ for $a_3 \neq t_2$. By induction hypothesis, there exists a 3^*-fan $A(s \to \{t_1, t_2, a_3\})$ of Q^0_{n-1}. Thus, the following paths $P(s, t_1), P(s, t_2)$, and $\langle s \xrightarrow{P(s,a_3)} a_3, \phi(a_3) \xrightarrow{P(\phi(a_3),t_3)} t_3\rangle$ form a 3^*-fan $A(s \to T)$ of $Q_n - F_a$.

Case 1.11 $|T^1| \geq 2$ and $f^1_a + |T^1| = n - 1$.
Let $(\phi(a_i), t_i)$ be edges of Q^1_{n-1} for $4 \leq i \leq k$. Applying Lemma 1, we can obtain a Hamiltonian cycle $\langle t_3, \phi(a_2) \xrightarrow{P(\phi(a_2),t_2)} t_2, \phi(a_3) \xrightarrow{P(\phi(a_3),t_3)} t_3\rangle$ of $Q^1_{n-1} - F^1_a - \{t_i, \phi(a_i)|$ for $4 \leq i \leq k$. By induction hypothesis, there exists a k^*-fan $A(s \to \{t_1, a_2, a_3, \cdots, a_k\})$ of Q^0_{n-1}. Thus, the following paths $P(s, t_1), \langle s \xrightarrow{P(s,a_i)} a_i, \phi(a_i), t_i\rangle$, and $\langle s \xrightarrow{P(s,a_j)} a_j, \phi(a_j) \xrightarrow{P(\phi(a_j),t_j)} t_j\rangle$ form a k^*-fan $A(s \to T)$ of $Q_n - F_a$ for $i = 2, 3$ and $4 \leq j \leq k$.

Case 2 $t_1 \in V^1$ and $f^1_a = 0$.
Without loss of generality, we can assume that $t_1 \neq \phi(s)$.
Case 2.1 $f^0_a = n - 3$ and $|T^1| = 1$.
Applying Lemma 1, we can construct a Hamiltonian cycle $\langle s \xrightarrow{P(s,t_2)} t_2, a_2 \xrightarrow{P(a_2,t_3)} t_3, a_3 \xrightarrow{P(a_3,s)} s\rangle$ of $Q^0_{n-1} - F^0_a$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s), \phi(a_2))$ and $P(\phi(a_3), t_1)$. Thus, these three paths $\langle s \xrightarrow{P(s,a_3)} a_3, \phi(a_3) \xrightarrow{P(\phi(a_3),t_1)} t_1\rangle, P(s, t_2), \langle s, \phi(s) \xrightarrow{P(\phi(s),\phi(a_2))} \phi(a_2), a_2 \xrightarrow{P(a_2,t_3)} t_3\rangle$ form a 3^*-fan $A(s \to T)$ of $Q_n - F_a$.

Case 2.2 $f^0_a = n - 3$ and $|T^1| = 2$.
Without loss of generality, we can assume that $t_2 \in V^1$. Applying Lemma 1, we can construct a Hamiltonian cycle $\langle s \xrightarrow{P(s,t_3)} t_3, a \xrightarrow{P(a,s)} s\rangle$ of $Q^0_{n-1} - F^0_a$ for $\phi(a) \neq t_2$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s), t_2)$ and $P(\phi(a), t_1)$. Thus, these three paths $\langle s \xrightarrow{P(s,a)} a, \phi(a) \xrightarrow{P(\phi(a),t_1)} t_1\rangle, \langle s, \phi(s) \xrightarrow{P(\phi(s),t_2)} t_2\rangle, P(s, t_3)$ form a 3^*-fan $A(s \to T)$ of $Q_n - F_a$.

Case 2.3 $f^0_a = n - 3$ and $|T^1| = 3$.
Let $(\phi(a_2), t_2)$ be an edge of Q^1_{n-1} for $a_2 \notin (F_a \cup \{s\})$. Applying Lemma 1, we can construct a Hamiltonian cycle $\langle s \xrightarrow{P(s,a_2)} a_2, a_1 \xrightarrow{P(a_1,s)} s\rangle$ of $Q^0_{n-1} - F^0_a$ for $\phi(a_1) \neq t_3$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s), t_3)$ and $P(\phi(a_1), t_1)$. Thus, these three paths $\langle s \xrightarrow{P(s,a_1)} a_1, \phi(a_1) \xrightarrow{P(\phi(a_1),t_1)} t_1\rangle, \langle s \xrightarrow{P(s,a_2)} a_2, \phi(a_2), t_2\rangle$, and $\langle s, \phi(s) \xrightarrow{P(\phi(s),t_3)} t_3\rangle$ form a 3^*-fan $A(s \to T)$ of $Q_n - F_a$.

Case 2.4 $f^0_a \leq n - 4$ and $|T^1| = 1$.
Let a be a white node in Q^0_{n-1}. By induction hypothesis, there exists a $(k-1)^*$-fan $A(s \to \{a, t_2, \cdots, t_{k-1}\})$ of $Q^0_{n-1} - F^0_a$. Without loss of generality, we can assume that t_k is on the path $P(s, t_{k-1})$. We can denote $P(s, t_{k-1})$ as $\langle s \xrightarrow{P(s,t_k)} t_k, b \xrightarrow{P(b,t_{k-1})} t_{k-1}\rangle$. Applying Lemma 3, we can obtain two spanning disjoint paths $P(\phi(s), \phi(b))$ and $P(\phi(a), t_1)$ of Q^1_{n-1}. Thus, the following paths $\langle s \xrightarrow{P(s,a)} a, \phi(a) \xrightarrow{P(\phi(a),t_1)} t_1\rangle, P(s, t_i), \langle s, \phi(s) \xrightarrow{P(\phi(s),\phi(b))} \phi(b), b \xrightarrow{P(b,t_{k-1})} t_{k-1}\rangle$, and $P(s, t_k)$ form a k^*-fan $A(s \to T)$ of $Q_n - F_a$ for $2 \leq i \leq k-2$.

Case 2.5 $f_a^0 \le n-4$ and $2 \le |T^1| \le n-2$.
Without loss of generality, we can assume that $t_2, t_3, \cdots, t_j \in V^1$. Let $(\phi(a_i), t_i)$ be edges of Q_{n-1}^1 such that $a_i \notin (F_a^0 \cup \{t_{j+1}, \cdots, t_k\})$ for $1 \le i \le j-1$. By induction hypothesis, there exists a $(k-1)^*$-fan $A(s \rightarrow \{a_1, a_2, \cdots, a_{j-1}, t_{j+1}, \cdots, t_k\})$ of $Q_{n-1}^0 - F_a^0$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(a_{j-1}), t_{j-1})$ and $P(\phi(s), t_j)$ of $Q_{n-1}^1 - \{t_i, \phi(a_i)| \text{ for } 1 \le i \le j-2\}$. Thus, the following paths $\langle s \overset{P(s,a_i)}{\longrightarrow} a_i, \phi(a_i), t_i\rangle, \langle s \overset{P(s,a_{j-1})}{\longrightarrow} a_{j-1}, \phi(a_{j-1}) \overset{P(\phi(a_{j-1}),t_{j-1})}{\longrightarrow} t_{j-1}\rangle$, $\langle s, \phi(s) \overset{P(\phi(s),t_j)}{\longrightarrow} t_j\rangle$ and $P(s, t_m)$ form a k^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$ for $1 \le i \le j-2$ and $j+1 \le m \le k$.

Case 2.6 $f_a^0 = 1$ and $|T^1| = n-1$.
Let x be a black node of V^1 such that the distance of x and the black faulty node is more than 2. By induction hypothesis, there exists a $(n-1)^*$-fan $A(x \rightarrow T)$ of Q_{n-1}^1. Without loss of generality, we can assume that $\phi(s)$ is on the path $P(x, t_1)$. Without loss of generality, we can assume that $P(x, t_1) = \langle x \overset{P(x,\phi(a_{n-1}))}{\longrightarrow} \phi(a_{n-1}), \phi(s) \overset{P(\phi(s),t_1)}{\longrightarrow} t_1\rangle$ for $a_{n-1} \notin F_a^0$. We can denote $P(x, t_i)$ as $\langle x, \phi(a_i) \overset{P(\phi(a_i),t_i)}{\longrightarrow} t_i\rangle$ for $2 \le i \le n-2$. By induction hypothesis, there exists a $(n-2)^*$-fan $A(s \rightarrow \{a_2, a_3, \cdots, a_{n-1}\})$ of $Q_{n-1}^0 - F_a^0$. Thus, the following paths $\langle s, \phi(s) \overset{P(\phi(s),t_1)}{\longrightarrow} t_1\rangle, \langle s \overset{P(s,a_i)}{\longrightarrow} a_i, \phi(a_i) \overset{P(\phi(a_i),t_i)}{\longrightarrow} t_i\rangle$ and $\langle s \overset{P(s,a_{n-1})}{\longrightarrow} a_{n-1}, \phi(a_{n-1}) \overset{P(\phi(a_{n-1}),x)}{\longrightarrow} x \overset{P(x,t_{n-1})}{\longrightarrow} t_{n-1}\rangle$ form a k^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$ for $1 \le i \le j-2$.

Case 3 $t_1 \in V^1$ and $f_a^1 \ge 1$.
Case 3.1 $f_a^1 + |T^1| = n$.
Let $(\phi(a_i), t_i)$ be edges of Q_{n-1}^1 for $4 \le i \le k$. Applying Lemma 1, we can obtain a Hamiltonian cycle $\langle t_1, \phi(a_3) \overset{P(\phi(a_3),t_3)}{\longrightarrow} t_3, \phi(a_2) \overset{P(\phi(a_2),t_2)}{\longrightarrow} t_2, \phi(a_1) \overset{P(\phi(a_1),t_1)}{\longrightarrow} t_1\rangle$ of $Q_{n-1}^1 - F_a^1 - \{\phi(a_i), t_i| \text{ for } 4 \le i \le k\}$. By induction hypothesis, there exists a k^*-fan $A(s \rightarrow \{a_1, a_2, \cdots, a_k\})$ of Q_{n-1}^0. Thus, the following paths $\langle s \overset{P(s,a_i)}{\longrightarrow} a_i, \phi(a_i) \overset{P(\phi(a_i),t_i)}{\longrightarrow} t_i\rangle, \langle s \overset{P(s,a_m)}{\longrightarrow} a_m, \phi(a_m), t_m\rangle$ form a k^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$ for $1 \le i \le 3$ and $4 \le m \le k$.

Case 3.2 $f_a^1 + |T^1| \le n-1$ and $f_a^1 \le n-4$.
Without loss of generality, we can assume that $t_2, t_3, \cdots, t_j \in V^1$. Let b be a white node of Q_{n-1}^0 such that $|N(b) \cap (F_a \cup \{t_{j+1}, \cdots, t_k\})| \le 1$ where $N(b)$ is the vertex set of the neighbor of b. By induction hypothesis, there exists a j^*-fan $A(\phi(b) \rightarrow \{t_1, t_2, \cdots, t_j\})$ of $Q_{n-1}^1 - F_a^1$. We can denote $(\phi(b) \rightarrow \{t_1, t_2, \cdots, t_j\})$ as $\langle \phi(b), \phi(a_i) \overset{P(\phi(a_i),t_i)}{\longrightarrow} t_i\rangle$ for $1 \le i \le j$. Without loss of generality, we can assume that $a_2, \cdots, a_j \notin (F_a \cup \{t_{j+1}, \cdots, t_k\})$. By induction hypothesis, there exists a k^*-fan $A(s \rightarrow \{b, a_2, \cdots, a_j, t_{j+1}, \cdots, t_k\})$ of $Q_{n-1}^0 - F_a^0$. Thus, the following paths $\langle s \overset{P(s,b)}{\longrightarrow} b, \phi(b) \overset{P(\phi(b),t_1)}{\longrightarrow} t_1\rangle, \langle s \overset{P(s,a_i)}{\longrightarrow} a_i, \phi(a_i) \overset{P(\phi(a_i),t_i)}{\longrightarrow} t_i\rangle$ and $P(s, t_m)$ form a k^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$ for $2 \le i \le j$ and $j+1 \le m \le k$.

Case 3.3 $f_a^1 = n-3$ and $|T^1| = 1$.
Applying Lemma 1, we can obtain a Hamiltonian cycle $\langle t_1, \phi(a) \overset{P(\phi(a),t_1)}{\longrightarrow} t_1\rangle$ of $Q_{n-1}^1 - F_a^1$. By induction hypothesis, there exists a 3^*-fan $A(s \rightarrow \{a, t_2, t_3\}$ of Q_{n-1}^0. Thus, the following paths $\langle s \overset{P(s,a)}{\longrightarrow} a, \phi(a) \overset{P(\phi(a),t_1)}{\longrightarrow} t_1\rangle, P(s, t_2), P(s, t_3)$ form a 3^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$.

Case 3.4 $f_a^1 = n-3$ and $|T^1| = 2$.
Without loss of generality, we can assume that $t_2 \in V^1$. Applying Lemma 1, we can obtain a Hamiltonian cycle $\langle t_1, \phi(a_2) \overset{P(\phi(a_2),t_2)}{\longrightarrow} t_2, \phi(a_1) \overset{P(\phi(a_1),t_1)}{\longrightarrow} t_1\rangle$ of $Q_{n-1}^1 - F_a^1$ for $\phi(a_2) \ne t_3$. By induction hypothesis, there exists a 3^*-fan $A(s \rightarrow \{a_1, a_2, t_3\}$ of Q_{n-1}^0. Thus, the following paths $\langle s \overset{P(s,a_i)}{\longrightarrow} a_i, \phi(a_i) \overset{P(\phi(a_i),t_i)}{\longrightarrow} t_i\rangle, P(s, t_3)$ form a 3^*-fan $A(s \rightarrow T)$ of $Q_n - F_a$ for $i = 1, 2$. □

III. THE ADJACENT VERTICES FAULT-TOLERANCE OF BIFANABILITY

The following lemma is proved in [5].

Lemma 4: Let $Q_n = \{V_b \cup V_w, E\}$. For $a \in V_b, b \in V_w$, the graph $Q_n - \{a, b\}$ is f_a-adjacency $(n - f_a - 3)$ edges Hamiltonian laceable for $f_a \le n-4$, $n \ge 4$.

Lemma 5: Let $s_1, t_1 \in V_w$ and $s_2, t_2 \in V_b$ be two pairs of fault-free vertices and F_a be the set with f_a pairs of adjacently faulty vertices of Q_n. There exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n - F_a$ for $f_a \le n-4, n \ge 4$.

Proof. By symmetry of hypercube, we can arrange every adjacently faulty nodes either in Q^0_{n-1} or Q^1_{n-1}. Let $T = \{s_1, t_1, s_2, t_2\}$ and T^i be the subset of vertices of T in Q^i_{n-1} for $i = 0, 1$. Without loss of generality, we can assume that $|T^0| \ge |T^1|$. We will prove this lemma in the following cases.

Case 1 $|T^0| = 4$ and $f^1_a = 0$.
Applying Lemma 3, we can construct two spanning disjoint paths $\langle s_1 \overset{P(s_1,b_1)}{\longrightarrow} b_1, w_1 \overset{P(w_1,t_2)}{\longrightarrow} t_2\rangle$ and $\langle s_2 \overset{P(s_2,b_2)}{\longrightarrow} b_2, w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ of $Q^0_{n-1} - F^0_a$. Applying Lemma 3, we can further construct two spanning disjoint paths $P(\phi(b_1), \phi(w_2))$ and $P(\phi(b_2), \phi(w_1))$ of Q^1_{n-1}. Thus, $\langle s_1 \overset{P(s_1,b_1)}{\longrightarrow} b_1, \phi(b_1) \overset{P(\phi(b_1),\phi(w_2))}{\longrightarrow} \phi(w_2), w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,b_2)}{\longrightarrow} b_2, \phi(b_2) \overset{P(\phi(b_2),\phi(w_1))}{\longrightarrow} \phi(w_1), w_1 \overset{P(w_1,t_2)}{\longrightarrow} t_2\rangle$ are two spanning disjoint paths of $Q_n - F_a$.

Case 2 $|T^0| = 4$ and $f^1_a \ge 1$.
Let (x_1, t_1) be an edge of Q^0_{n-1} for $\phi(x_1) \notin F^1_a$. Let $E(t_2) = \{(v, t_2) | \phi(v) \in F^1_a\}$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_2 \overset{P(s_2,t_2)}{\longrightarrow} t_2, x_2 \overset{P(x_2,s_1)}{\longrightarrow} s_1\rangle$ of $Q^0_{n-1} - F^0_a - \{x_1, t_1\} - E(t_2)$. Thus, $\langle s_1 \overset{P(s_1,x_2)}{\longrightarrow} x_2, \phi(x_2) \overset{P(\phi(x_2),\phi(x_1))}{\longrightarrow} \phi(x_1), x_1, t_1\rangle$ and $P(s_2, t_2)$ are two spanning disjoint paths of $Q_n - F_a$.

Case 3 $|T^0| = 3$.
Without loss of generality, we can assume that $t_2 \in V^1$. Let $E(t_1) = \{(v, t_1) | \phi(v) \in F^1_a\}$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \overset{P(s_1,t_1)}{\longrightarrow} t_1, x \overset{P(x,s_2)}{\longrightarrow} s_2\rangle$ of $Q^0_{n-1} - F^0_a - E(t_1)$. Applying Lemma 1, we can also construct a Hamiltonian path $P(\phi(x), t_2)$ of $Q^1_{n-1} - F^1_a$. Thus, $P(s_1, t_1)$ and $\langle s_2 \overset{P(s_2,x)}{\longrightarrow} x, \phi(x) \overset{P(\phi(x),t_2)}{\longrightarrow} t_2\rangle$ are two spanning disjoint paths of $Q_n - F_a$.

Case 4 $|T^0| = 2$.
Without loss of generality, we can assume that $s_1 \in V^0$ and $t_2 \in V^1$. Suppose $t_1 \in V^0$ and $s_2 \in V^1$. Let x_1 be a black node of Q^0_{n-1} for $x_1, \phi(x_1) \notin (F_a \cup T)$. Let $E(t_1) = \{(v, t_1) | \phi(v) \in F^1_a\}$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \overset{P(s_1,t_1)}{\longrightarrow} t_1, x_2 \overset{P(x_2,x_1)}{\longrightarrow} x_1\rangle$ of $Q^0_{n-1} - F^0_a - E(t_1)$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(x_1), t_2)$ and $P(s_2, \phi(x_2))$ of $Q^1_{n-1} - F^1_a$. Thus, $P(s_1, t_1)$ and $\langle s_2 \overset{P(s_2,\phi(x_2))}{\longrightarrow} \phi(x_2), x_2 \overset{P(x_2,x_1)}{\longrightarrow} x_1, \phi(x_1) \overset{P(\phi(x_1),t_2)}{\longrightarrow} t_2\rangle$ are two spanning disjoint paths of $Q_n - F_a$.
Suppose $s_2 \in V^0$ and $t_1 \in V^1$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \overset{P(s_1,w)}{\longrightarrow} w, b \overset{P(b,s_2)}{\longrightarrow} s_2\rangle$ of $Q^0_{n-1} - F^0_a$ for $w \in V_w, b \in V_b$ and $\phi(b), \phi(w) \notin (F^1_a \cup T^1)$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(b), t_2)$ and $P(\phi(w), t_1)$ of $Q^1_{n-1} - F^1_a$. Thus, $\langle s_1 \overset{P(s_1,w)}{\longrightarrow} w, \phi(w) \overset{P(\phi(w),t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,b)}{\longrightarrow} b, \phi(b) \overset{P(\phi(b),t_2)}{\longrightarrow} t_2\rangle$ are two spanning disjoint paths of $Q_n - F_a$. □

In the following, we will prove the adjacent vertices fault-tolerance for bifanability of hypercube.

Theorem 2: The graph $Q_n - F_a$ is $(n - |F_a| - 1)$-bifanable graph if $|F_a| \le n-3$ for $n \ge 3$.

Proof: We will prove this theorem by induction on n. Since Q_n has property 2H, this theorem is true for $n = 3$. We can verify this theorem for $n = 4$ by brute force. Applying Lemma 3, we can obtain that this theorem holds if $|F_a| = n-3$. In the follows, we will assume that $|F_a| \le n-4$ and $n \ge 5$. By symmetry of hypercube, we can assume that every pair of adjacently faulty vertices is either in Q^0_{n-1} or Q^1_{n-1}. Let s_1, s_2 be the source vertices for $s_1 \in V_b$ and $s_2 \in V_w$. Let $T_1 = \{t^1_1, t^2_1, \cdots, t^{k_1}_1\}$ and $T_2 = \{t^1_2, t^2_2, \cdots, t^{k_2}_2\}$ be the sets of end vertices of s_1 and s_2, respectively. And let the vertices $t^1_1, t^i_2 \in V_w$ and $t^1_2, t^j_1 \in V_w$ for $2 \le i \le k_1, 2 \le j \le k_2$. Let T^r_k be the set of end vertices of s_k in Q^r for $r = 0, 1$ and $k = 1, 2$. Let $T^r = T^r_1 \cup T^r_2$ for $r = 0, 1$. Without loss of generality, we can assume that $k_1 \ge k_2$. Thus, $k_1 \ge 2$. We will prove the induction step with the following cases.

Case 1 $s_1, s_2 \in V^0$ or $s_1, s_2 \in V^1$.
Without loss of generality, we can assume that $s_1, s_2 \in V^0$.

Case 1.1 $|T^1| = 0$ and $|F^1_a| = 0$.
Let $T'_1 = T_1 - \{t^{k_1}_1\}$. By induction hypothesis, we can construct $(n\text{-}2\text{-}f_a)^*$-bifan $B(s_1 \to T'_1, s_2 \to T_2)$ of $Q^0_{n-1} - F^0_a$. Suppose the vertex $t^{k_1}_1$ is on the path $P(s_1, t^i_1)$ for some $1 \le i \le k_1 - 1$. We can denote the path $P(s_1, t^i_1)$ as $\langle s_1 \overset{P(s_1,t^{k_1}_1)}{\longrightarrow} t^{k_1}_1, x \overset{P(x,t^i_1)}{\longrightarrow} t^i_1\rangle$.

Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(s_1),\phi(x))$ of Q^1_{n-1}. We can construct the $B(s_1 \to T_1, s_2 \to T_2)$ from $B(s_1 \to T'_1, s_2 \to T_2)$ by replacing $P(s_1, t^i_1)$ with $\langle s_1, \phi(s_1) \overset{P(\phi(s_1),\phi(x))}{\longrightarrow} \phi(x), x \overset{P(x,t^i_1)}{\longrightarrow} t^i_1\rangle$ and $P(s_1, t^{k_1}_1)$ of $Q_n - F_a$. Suppose the vertex $t^{k_1}_1$ is on the path $P(s_2, t^i_2)$ for some $1 \le i \le k_2$. We can denote the path $P(s_2, t^i_2)$ as $\langle s_2 \overset{P(s_2,x_1)}{\longrightarrow} x_1, t^{k_1}_1, x_2 \overset{P(x_2,t^i_2)}{\longrightarrow} t^i_2\rangle$. Applying Lemma 5, we can construct two Hamiltonian paths $P(\phi(s_1), \phi(t^{k_1}_1))$ and $P(\phi(x_1), \phi(x_2))$ of Q^1_{n-1}. We can construct the $B(s_1 \to T_1, s_2 \to T_2)$ from $B(s_1 \to T'_1, s_2 \to T_2)$ by replacing $P(s_2, t^i_2)$ with $\langle s_1, \phi(s_1) \overset{P(\phi(s_1),\phi(t^{k_1}_1))}{\longrightarrow} \phi(t^{k_1}_1), t^{k_1}_1\rangle$ and $\langle s_2 \overset{P(s_2,x_1)}{\longrightarrow} x_1, \phi(x_1) \overset{P(\phi(x_1),\phi(x_2))}{\longrightarrow} \phi(x_2), x_2 \overset{P(x_2,t^i_2)}{\longrightarrow} t^i_2\rangle$ of $Q_n - F_a$.

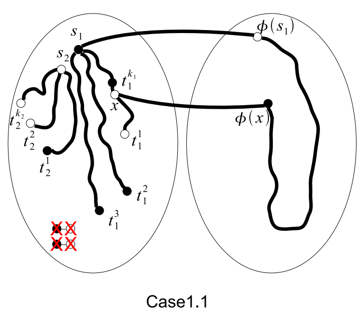

Fig. 3. Illustration of **Case 2.1**

Case 1.2 $|T^1| = 0$ and $|F^1_a| \ge 1$.
By induction hypothesis, we can construct the $B(s_1 \to T_1, s_2 \to T_2)$ of $Q^0_{n-1} - F^0_a$. Without loss of generality, we can assume that the path $P(s_1, t^1_1)$ can be denoted as $\langle s_1 \overset{P(s_1,x_1)}{\longrightarrow} x_1, x_2 \overset{P(x_2,t^1_1)}{\longrightarrow} t^1_1\rangle$ for $\phi(x_1), \phi(x_2) \notin F^1_a$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(x_1), \phi(x_2))$ of $Q^1_{n-1} - F^1_a$. Thus, we can construct $B(s_1 \to T_1, s_2 \to T_2)$ by replacing $P(s_1, t^1_1)$ with $\langle s_1 \overset{P(s_1,x_1)}{\longrightarrow} x_1, \phi(x_1) \overset{P(\phi(x_1),\phi(x_2))}{\longrightarrow} \phi(x_2), x_2 \overset{P(x_2,t^1_1)}{\longrightarrow} t^1_1\rangle$.

Case 1.3 $|T^1| \ge 1$ and $|F^1_a| \ge 1$.

Case 1.3.1 $|T^1| = 1$.
Without loss of generality, we can assume that $t^1_1 \in Q^1_{n-1}$. Let $x_1 \in Q^0_{n-1}$ be a white vertex such that $x_1, \phi(x_1) \notin (F_a \cup T_1 \cup T_2)$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(x_1), t^1_1)$ of $Q^1_{n-1} - F^1_a$. By induction hypothesis, we can construct $B(s_1 \to \{x_1, t^2_1, \cdots, t^{k_1}_1\}, s_2 \to T_2)$ of $Q^0_{n-1} - F^0_a$. Thus, we can construct $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$ from $B(s_1 \to \{x_1, t^2_1, \cdots, t^{k_1}_1\}, s_2 \to T_2)$ with replacing the path $P(s_1, x_1)$ by the path $\langle s_1 \overset{P(s_1,x_1)}{\longrightarrow} x_1, \phi(x_1) \overset{P(\phi(x_1),t^1_1)}{\longrightarrow} t^1_1\rangle$.

Case 1.3.2 $|T^1| \ge 2$ and $|T^1| + |F^1_a| \le n-2$.
Without loss of generality, we can assume that $t^1_1, t^2_1, \cdots, t^{i_1}_1, t^1_2, t^2_2, \cdots, t^{i_2}_2 \in V^1$ and $i_1 \ge 2$. Let $(\phi(x^{j_1}_1), t^{j_1}_1)$ and $(\phi(x^{j_2}_2), t^{j_2}_2)$ be edges of Q^1_{n-1} such that $x^{j_1}_1, x^{j_2}_2, \phi(x^{j_1}_1), \phi(x^{j_2}_2) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\})$ for $1 \le j_1 \le i_1, 1 \le j_2 \le i_2$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(x^1_1), t^1_1)$ and $P(\phi(x^2_1), t^2_1)$. By induction hypothesis, we can construct $B(s_1 \to \{x^1_1, \cdots, x^{i_1}_1, t^{i_1+1}_1, \cdots, t^{k_1}_1\}, s_2 \to \{x^1_2, \cdots, x^{i_2}_2, t^{i_2+1}_2, \cdots, t^{k_2}_2\})$ of $Q^0_{n-1} - F^0_a$. Thus, $\langle s_1 \overset{P(s_1,x^m_1)}{\longrightarrow} x^m_1, \phi(x^m_1) \overset{P(\phi(x^m_1),t^m_1)}{\longrightarrow} t^m_1\rangle, \langle s_1 \overset{P(s_1,x^{j_1}_1)}{\longrightarrow} x^{j_1}_1, \phi(x^{j_1}_1), t^{j_1}_1\rangle, P(s_1, t^{r_1}_1), \langle s_2 \overset{P(s_2,x^{j_2}_2)}{\longrightarrow} x^{j_2}_2, \phi(x^{j_2}_2), t^{j_2}_2\rangle, P(s_2, t^{r_2}_2)$ form the (n-1-f_a)*-bifan of $Q_n - F_a$ for $1 \le m \le 2, 3 \le j_1 \le i_1, i_1+1 \le r_1 \le k_1, 1 \le j_2 \le i_2, i_2+1 \le r_2 \le k_2$.

Case 1.3.3 $|T^1| \ge 2$ and $|T^1| + |F^1_a| = n-1$.
Let $(\phi(x^{j_1}_1), t^{j_1}_1)$ and $(\phi(x^{j_2}_2), t^{j_2}_2)$ be edges of Q^1_{n-1} such that $x^{j_1}_1, x^{j_2}_2, \phi(x^{j_1}_1), \phi(x^{j_2}_2) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\})$ for $1 \le j_1 \le k_1, 1 \le j_2 \le k_2$. Applying Lemma 3, we can construct two spanning disjoint paths $\langle t^2_1 \overset{P(t^2_1,\phi(x^2_1))}{\longrightarrow} \phi(x^2_1), \phi(x^1_1) \overset{P(\phi(x^1_1),t^1_1)}{\longrightarrow} t^1_1\rangle$ and $P(\phi(x^3_1), t^3_1)$ of $Q^1_{n-1} - F^1_a - \{\phi(x^{j_1}), t^{j_1}_1, \phi(x^{j_2}_2), t^{j_2}_2 |$ for $4 \le j_1 \le k_1$ and $1 \le j_2 \le k_2\}$. By induction hypothesis, we can construct $B(s_1 \to \{x^1_1, x^2_1, \cdots, x^{k_1}_1\}, s_2 \to \{x^1_2, x^2_2, \cdots, x^{k_2}_2\})$ of Q^0_{n-1}. Thus, $\langle s_1 \overset{P(s_1,x^m_1)}{\longrightarrow} x^m_1, \phi(x^m_1) \overset{P(\phi(x^m_1),t^m_1)}{\longrightarrow} t^m_1\rangle$, $\langle s_1 \overset{P(s_1,x^{j_1}_1)}{\longrightarrow} x^{j_1}_1, \phi(x^{j_1}_1), t^{j_1}_1\rangle, \langle s_2 \overset{P(s_2,x^{j_2}_2)}{\longrightarrow} x^{j_2}_2, \phi(x^{j_2}_2), t^{j_2}_2\rangle$ form the (n-1-f_a)*-bifan of $Q_n - F_a$ for $1 \le m \le 3, 4 \le j_1 \le k_1, 1 \le j_2 \le k_2$.

Case 1.4 $|T^1| \ge 1$ and $|F^1_a| = 0$.

Case 1.4.1 $t^1_1 \in V^1$ or $t^1_2 \in V^1$.
Without loss of generality, we can assume that $t^1_1 \in V^1$. Suppose that $k_2 \ge 2$. By induction hypothesis, we can construct $B(s_1 \to \{t^2_2, t^2_1, \cdots, t^{k_1}_1\}, s_2 \to \{t^1_2, t^3_2, \cdots, x^{k_2}_2\})$ of $Q^0_{n-1} - F^0_a$. We can denote $P(s_1, t^2_2)$ as $\langle s_1, x_1 \overset{P(x_1,t^2_2)}{\longrightarrow} t^2_2\rangle$. Applying Lemma 5, we can construct two span-

ning disjoint paths $P(\phi(s_1), t_1^1)$, $P(\phi(s_2), \phi(x_1))$ of Q_{n-1}^1. Thus, we can construct $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of $Q_n - F_a$ from $B(s_1 \rightarrow \{t_2^2, t_1^2, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{t_2^1, t_2^3, \cdots, x_2^{k_2}\})$ with replacing $P(s_1, t_2^2)$ by the paths $\langle s_1, \phi(s_1) \stackrel{P(\phi(s_1),t_1^1)}{\longrightarrow} t_1^1\rangle$ and $\langle s_2, \phi(s_2) \stackrel{P(\phi(s_2),\phi(x_1))}{\longrightarrow} \phi(x_1), x_1 \stackrel{P(x_1,t_2^2)}{\longrightarrow} t_2^2\rangle$.

Suppose that $k_2 = 1$. Applying Theorem 1, we can construct a fan $A(s_1 \rightarrow \{t_1^2, \cdots, t_1^{k_1}, s_2, t_2^1\})$ of $Q_{n-1}^0 - F_a$. We can denote the paths $P(s_2, s_1)$ and $P(s_1, t_2^1)$ as $\langle s_2 \stackrel{P(s_2,x_1)}{\longrightarrow} x_1, s_1\rangle$ and $\langle s_1, x_2 \stackrel{P(x_2,t_2^1)}{\longrightarrow} t_2^1\rangle$, respectively. Applying Lemma 5, we can construct two spanning disjoint paths $P(\phi(s_1), t_1^1)$ and $P(\phi(x_1), \phi(x_2))$ of Q_{n-1}^1. Thus, we can construct the $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of $Q_n - F_a$ from $A(s_1 \rightarrow \{t_1^2, \cdots, t_1^{k_1}, s_2, t_2^1\})$ by removing the paths $P(s_1, s_2)$ and $P(s_1, t_2^1)$ and adding the paths $\langle s_1, \phi(s_1) \stackrel{P(\phi(s_1),t_1^1)}{\longrightarrow} t_1^1\rangle$ and $\langle s_2 \stackrel{P(s_2,x_1)}{\longrightarrow} x_1, \phi(x_1) \stackrel{P(\phi(x_1),\phi(x_2))}{\longrightarrow} \phi(x_2), x_2 \stackrel{P(x_2,t_2^1)}{\longrightarrow} t_2^1\rangle$.

Case 1.4.2 $t_1^1, t_2^1 \in Q_{n-1}^1$ and $|F_a| \geq 1$.

Let b_1 and w_1 be a pair of adjacently faulty vertices of Q_{n-1}^0. By induction hypothesis, we can construct $B(s_1 \rightarrow \{w_1, t_1^2, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{b_1, t_2^2, \cdots, t_2^{k_2}\})$ of $Q_{n-1}^0 - (F_a^0 - \{b_1, w_1\})$. We can denote the paths $P(s_1, w_1)$ and $P(s_2, b_1)$ as $\langle s_1 \stackrel{P(s_1,x_1)}{\longrightarrow} x_1, w_1\rangle$ and $\langle s_2 \stackrel{P(s_2,x_2)}{\longrightarrow} x_2, b_2\rangle$, respectively. Suppose that $\{\phi(x_1), \phi(x_2)\} \cap \{t_1^1, t_2^1\} = \emptyset$. Applying Lemma 5, we can construct two spanning disjoint paths $P(\phi(x_1), t_1^1)$ and $P(\phi(x_2), t_2^1)$ of Q_{n-1}^1. Thus, we can construct the $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of $Q_n - F_a$ from $B(s_1 \rightarrow \{w_1, t_1^2, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{b_1, t_2^2, \cdots, t_2^{k_2}\})$ by removing the paths $P(s_1, w_1)$ and $P(s_2, b_1)$ and adding the paths $\langle s_1 \stackrel{P(s_1,x_1)}{\longrightarrow} x_1, \phi(x_1) \stackrel{P(\phi(x_1),t_1^1)}{\longrightarrow} t_1^1\rangle$ and $\langle s_2 \stackrel{P(s_2,x_2)}{\longrightarrow} x_2, \phi(x_2) \stackrel{P(\phi(x_2),t_2^1)}{\longrightarrow} t_2^1\rangle$.

Suppose that $|\{\phi(x_1), \phi(x_2)\} \cap \{t_1^1, t_2^1\}| = 1$. Without loss of generality, we can assume that $\phi(x_2) = t_2^1$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(x_1), t_1^1)$ of $Q_{n-1}^1 - \{t_2^1\}$. Thus, we can construct the $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of $Q_n - F_a$ from $B(s_1 \rightarrow \{w_1, t_1^2, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{b_1, t_2^2, \cdots, t_2^{k_2}\})$ by removing the paths $P(s_1, w_1)$ and $P(s_2, b_1)$ and adding the paths $\langle s_1 \stackrel{P(s_1,x_1)}{\longrightarrow} x_1, \phi(x_1) \stackrel{P(\phi(x_1),t_1^1)}{\longrightarrow} t_1^1\rangle$ and $\langle s_2 \stackrel{P(s_2,x_2)}{\longrightarrow} x_2, t_2^1\rangle$.

Suppose that $\{\phi(x_1), \phi(x_2)\} = \{t_1^1, t_2^1\}$. That is, $\phi(x_1) = t_1^1$ and $\phi(x_2) = t_2^1$. We denote $P(s_1, t_1^2)$ as $\langle s_1, y_1 \stackrel{P(y_1,t_1^2)}{\longrightarrow} t_1^2\rangle$. Applying Lemma 4, we can construct a Hamiltonian path $P(\phi(s_1), \phi(y_1))$ of $Q_{n-1}^1 - \{t_1^1, t_1^2\}$. Thus, we can construct the $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of $Q_n - F_a$ from $B(s_1 \rightarrow \{w_1, t_1^2, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{b_1, t_2^2, \cdots, t_2^{k_2}\})$ by removing the paths $P(s_1, w_1)$, $P(s_2, b_1)$, $P(s_1, t_1^2)$ and adding the paths $\langle s_1 \stackrel{P(s_1,x_1)}{\longrightarrow} x_1, t_1^1\rangle$, $\langle s_2 \stackrel{P(s_2,x_2)}{\longrightarrow} x_2, t_2^1\rangle$, $\langle s_1, \phi(s_1) \stackrel{P(\phi(s_1),\phi(y_1))}{\longrightarrow} \phi(y_1), y_1 \stackrel{P(y_1,t_1^2)}{\longrightarrow} t_1^2\rangle$.

Case 1.4.3 $t_1^1, t_2^1 \in Q_{n-1}^1$ and $|F_a| = 0$.

Suppose that $n = 5$ and $k_1 = k_2 = 2$. Applying Lemma 5, we can construct two spanning disjoint paths $P(s_1, t_1^2), P(s_2, t_2^2)$ of Q_{n-1}^0 and two spanning disjoint paths $P(\phi(s_1), t_1^1), P(\phi(s_2), t_2^1)$ of Q_{n-1}^1. Thus, the following paths $\langle s_1, \phi(s_1) \stackrel{P(\phi(s_1),t_1^1)}{\longrightarrow} t_1^1\rangle, P(s_1, t_1^2)$, and $\langle s_2, \phi(s_2) \stackrel{P(\phi(s_2),t_2^1)}{\longrightarrow} t_2^1\rangle, P(s_2, t_2^2)$ form the 4^*-bifan of Q_n. Suppose that $n = 5$ and $k_1 = 3, k_2 = 1$. Applying Lemma 1, we can construct a Hamiltonian path $\langle t_1^2 \stackrel{P(t_1^2,s_1)}{\longrightarrow} s_1 \stackrel{P(s_1,t_1^3)}{\longrightarrow} t_1^3\rangle$ of $Q_{n-1}^0 - \{s_2\}$. Applying Lemma 5, we can construct two spanning disjoint paths $P(\phi(s_1), t_1^1), P(\phi(s_2), t_2^1)$ of Q_{n-1}^1. Thus, the following paths $\langle s_1, \phi(s_1) \stackrel{P(\phi(s_1),t_1^1)}{\longrightarrow} t_1^1\rangle, P(s_1, t_1^2), P(s_1, t_1^3)$ and $\langle s_2, \phi(s_2) \stackrel{P(\phi(s_2),t_2^1)}{\longrightarrow} t_2^1\rangle$ form the 4^*-bifan of Q_n.

Suppose that $n \geq 6$. Let $(\phi(x_1^1), t_1^1)$ be an edge of Q_{n-1}^1 for $x_1^1 \notin (T_2 \cup \{s_2\})$. By induction hypothesis, we can construct $B(s_1 \rightarrow \{x_1^1, t_1^3, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{t_1^2, t_2^2, \cdots, t_2^{k_2}\})$ of Q_{n-1}^0. We can denote the path $P(s_2, t_1^2)$ as $\langle s_2, x_2 \stackrel{P(x_2,t_1^2)}{\longrightarrow} t_1^2\rangle$. Applying Lemma 5, we can construct two spanning disjoint paths $P(\phi(s_1), \phi(x_2))$ and $P(\phi(s_2), t_2^1)$ of $Q_{n-1}^1 - \{t_1^1, \phi(x_1)\}$. Thus, we can construct the $B(s_1 \rightarrow T_1, s_2 \rightarrow T_2)$ of $Q_n - F_a$ from $B(s_1 \rightarrow \{x_1^1, t_1^3, \cdots, t_1^{k_1}\}, s_2 \rightarrow \{t_1^2, t_2^2, \cdots, t_2^{k_2}\})$ by removing the paths $P(s_1, x_1^1)$ and $P(s_2, t_1^2)$ and adding the paths $\langle s_1 \stackrel{P(s_1,x_1)}{\longrightarrow} x_1, t_1^1\rangle, \langle s_1, \phi(s_1) \stackrel{P(\phi(s_1),\phi(x_2))}{\longrightarrow} \phi(x_2), x_2 \stackrel{P(x_2,t_1^2)}{\longrightarrow} t_1^2\rangle$ and $\langle s_2, \phi(s_2) \stackrel{P(\phi(s_2),t_2^1)}{\longrightarrow} t_2^1\rangle$.

Case 1.4.4 $|(T_1 \cup T_2 - \{t_1^1, t_2^1\}) \cap T^1| \geq 1$ and $|T^1| \leq n - 3$.

Without loss of generality, we can assume that $t_1^1, \cdots, t_1^{j_1}, t_2^1, \cdots, t_2^{j_2} \in V^1$ for $j_1 \geq 2$. Let $(\phi(x_1^{i_1}), t_1^{i_1})$ and $(\phi(x_2^{i_2}), t_2^{i_2})$ be edges of Q_{n-1}^1

such that $\phi(x_1^{i_1}), x_1^{i_1}, \phi(x_2^{i_2}), x_2^{i_2} \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\})$ for $1 \le i_1 \le j_1 - 1, 1 \le i_2 \le j_2$. By induction hypothesis, we can construct $B(s_1 \to \{x_1^1, \cdots, x_1^{j_1-1}, t_1^{j_1+1}, \cdots, t_1^{k_1}\}, s_2 \to \{x_2^1, \cdots, x_2^{j_2},\ t_2^{j_2+1}, \cdots, t_2^{k_2}\})$ of $Q_{n-1}^0 - F_a^0$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(s_1), t_1^{j_1})$ of $Q_{n-1}^1 - \{t_1^{i_1}, \phi(x_1^{i_1}), t_2^{i_2}, \phi(x_2^{i_2})|$ for $1 \le i_1 \le j_1 - 1$ and $1 \le i_2 \le j_2\}$. Thus, the following paths $\langle s_1 \overset{P(s_1, x_1^{i_1})}{\longrightarrow} x_1^{i_1}, \phi(x_1^{i_1}), t_1^{i_1}\rangle$, $\langle s_1, \phi(s_1) \overset{P(\phi(s_1), t_1^{j_1})}{\longrightarrow} t_1^{j_1}\rangle$, $P(s_1, t_1^{i_2})$ and $\langle s_2 \overset{P(s_2, x_2^{r_1})}{\longrightarrow} x_2^{r_1}, \phi(x_2^{r_1}), t_2^{r_1}\rangle$, $P(s_2, t_r^{r_2})$ form the (n-1-f_a)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$ for $1 \le i_1 \le j_1 - 1, j_1 + 1 \le i_2 \le k_1, 1 \le r_1 \le j_2, j_2 + 1 \le r_2 \le k_2$.

Case 1.4.5 $|T^1| = n - 2$.
Suppose that $|F_a| = 1$. Let $(\phi(x_1^{i_1}), t_1^{i_1})$ and $(\phi(x_2^{i_2}), t_2^{i_2})$ be edges of Q_{n-1}^1 such that $x_1^{i_1}, x_2^{i_2} \notin (F_a \cup \{s_1, s_2\})$ for $1 \le i_1 \le k_1 - 1, 1 \le i_2 \le k_2$. By induction hypothesis, we can construct $B(s_1 \to \{x_1^1, \cdots, x_1^{k_1-1}\}, s_2 \to \{x_2^1, \cdots, x_2^{k_2}\})$ of $Q_{n-1}^0 - F_a^0$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(x_1^{k_1-1}), t_1^{k_1-1})$ and $P(\phi(s_1), t_1^{k_1})$ of $Q_{n-1}^1 - \{t_1^{i_1}, \phi(x_1^{i_1}), t_2^{i_2}, \phi(x_2^{i_2})|$ for $1 \le i_1 \le k_1 - 2$ and $1 \le i_2 \le k_2\}$. Thus, the following paths $\langle s_1 \overset{P(s_1, x_1^{i_1})}{\longrightarrow} x_1^{i_1}, \phi(x_1^{i_1}), t_1^{i_1}\rangle$, $\langle s_1 \overset{P(s_1, x_1^{k_1-1})}{\longrightarrow} x_1^{k_1-1}, \phi(x_1^{k_1-1}) \overset{P(\phi(x_1^{k_1-1}), t_1^{k_1-1})}{\longrightarrow} t_1^{k_1-1}\rangle$, $\langle s_1, \phi(s_1) \overset{P(\phi(s_1), t_1^{k_1})}{\longrightarrow} t_1^{k_1}\rangle$ and $\langle s_2 \overset{P(s_2, x_2^{i_2})}{\longrightarrow} x_2^{i_2}, \phi(x_2^{i_2}), t_2^{i_2}\rangle$ form the (n-2)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$ for $1 \le i_1 \le k_1 - 2, 1 \le i_2 \le k_2$.
The proof for $|F_a^1| = 0$ is similar as $|F_a^1| = 1$.

Case 1.4.6 $|T^1| = n - 1$.
Suppose that $|T_1| \ge 3$. Let $(\phi(x_1^{i_1}), t_1^{i_1})$ and $(\phi(x_2^{i_2}), t_2^{i_2})$ be edges of Q_{n-1}^1 such that $x_1^{i_1}, x_2^{i_2} \notin \{s_1, s_2\}$ for $4 \le i_1 \le k_1, 1 \le i_2 \le k_2$. Applying Lemma 3, we can construct two spanning disjoint paths $P(\phi(s_1), t_1^3)$ and $\langle t_1^1 \overset{P(t_1^1, \phi(x_1^1))}{\longrightarrow} \phi(x_1^1), \phi(x_1^2) \overset{P(\phi(x_1^2), t_1^2)}{\longrightarrow} t_1^2\rangle$ of $Q_{n-1}^1 - \{t_1^{i_1}, \phi(x_1^{i_1}), t_2^{i_2}, \phi(x_2^{i_2})|$ for $4 \le i_1 \le k_1$ and $1 \le i_2 \le k_2\}$ with $x_1^1, x_1^2 \notin \{s_1, s_2\}$. By induction hypothesis, we can construct (n-2)*-bifan $B(s_1 \to \{x_1^1, x_1^2, x_1^4, \cdots, x_1^{k_1-1}\}, s_2 \to \{x_2^1, \cdots, x_2^{k_2}\})$ of Q_{n-1}^0. Thus, the following paths $\langle s_1 \overset{P(s_1, x_1^{i_1})}{\longrightarrow} x_1^{i_1}, \phi(x_1^{i_1}) \overset{P(\phi(x_1^{i_1}), t_1^{i_1})}{\longrightarrow} t_1^{i_1}\rangle$, $\langle s_1, \phi(s_1) \overset{P(\phi(s_1), t_1^3)}{\longrightarrow} t_1^3\rangle$, $\langle s_1 \overset{P(s_1, x_1^{i_2})}{\longrightarrow} x_1^{i_2}, \phi(x_1^{i_2}), t_1^{i_2}\rangle$, and $\langle s_2 \overset{P(s_2, x_2^{i_3})}{\longrightarrow} x_2^{i_3}, \phi(x_2^{i_3}), t_2^{i_3}\rangle$ form the (n-1)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of Q_n for $1 \le i_1 \le 2, 4 \le i_2 \le k_1, 1 \le i_3 \le k_2$.
The proof for $|T^1| = 2$ is similar as $|T^1| \ge 3$.

Case 2 $s_1 \in V^i$ and $s_2 \in V^j$ for $i \ne j$.
Without loss of generality, we can assume that $s_1 \in V^0$ and $s_2 \in V^1$. We can also assume that $|T_1^1| + |F_a^1| \ge |T_2^0| + |F_a^0|$.
Case 2.1 $|T_2^0| + |F_a^0| \le |T_1^1| + |F_a^1| \le n - 4$.
Without loss of generality, we can assume that $t_1^1, \cdots, t_1^{j_1} \in V^1$ and $t_2^1, \cdots, t_2^{j_2} \in V^0$. Let $(\phi(x^{i_1}), t_1^{i_1})$ be edges of Q_{n-1}^1 such that $x^{i_1}, \phi(x^{i_1}) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\})$ for $1 \le i_1 \le j_1$. Let $(y^{i_2}, t_2^{i_2})$ be edges of Q_{n-1}^0 such that $y^{i_2}, \phi(y^{i_2}) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\} \cup \{x^{i_1}, \phi(x^{i_1})|$ for $1 \le i_1 \le j_1\})$ with $1 \le i_2 \le j_2$. Applying Theorem 1, we can construct a k_1^*-fan $A(s_1 \to \{x^1, \cdots, x^{j_1}, t_1^{j_1+1}, \cdots, t_1^{k_1}\})$ of $Q_{n-1}^0 - F_a^0 - \{t_2^{i_2}, y^{i_2}|$ for $1 \le i_2 \le j_2\}$ and a k_2^*-fan $A(s_2 \to \{\phi(y^1), \cdots, \phi(y^{j_2}), t_2^{j_2+1}, \cdots, t_2^{k_2}\})$ of $Q_{n-1}^1 - F_a^1 - \{t_1^{i_1}, \phi(x^{i_1})|$ for $1 \le i_1 \le j_1\}$. Thus, the following paths $\langle s_1 \overset{P(s_1, x^{i_1})}{\longrightarrow} x^{i_1}, \phi(x^{i_1}), t_1^{i_1}\rangle$, $P(s_1, t_1^{i_2})$ and $\langle s_2 \overset{P(s_2, \phi(y^{i_3}))}{\longrightarrow} \phi(y^{i_3}), y^{i_3}, t_2^{i_3}\rangle$, $P(s_2, t_2^{i_4})$ form the (n-1-f_a)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$ for $1 \le i_1 \le j_1, j_1 + 1 \le i_2 \le k_1, 1 \le i_3 \le j_2, j_2 + 1 \le i_4 \le k_2$.
Case 2.2 $|T_1^1| + |F_a^1| = n - 3$.
Case 2.2.1 $|T_2| = 2$.
Suppose that $t_2^1, t_2^2 \in V^0$ and $d(s_2, t_2^2) \ge 3$. Let $(y^1, t_2^1), (y^2, t_2^2)$ be edges of Q_{n-1}^0 for $\phi(y^1), \phi(y^2) \notin (T^1 \cup F_a^1 \cup \{s_2\})$. Let $(\phi(x^i), t_1^i)$ be an edge of Q_{n-1}^1 for $x^i \notin \{s_1, t_2^1, t_2^2, y^1, y^2\}, 2 \le i \le k_1$. Applying Theorem 1, we can construct a 3*-fan $A(s_2 \to \{\phi(y^1), \phi(y^2), t_1^1\})$ of $Q_{n-1}^1 - F_a^1 - \{t_1^i, \phi(x^i)|2 \le i \le k_1\}$. We denote the three paths of $A(s_2 \to \{\phi(y^1), \phi(y^2), t_1^1\})$ as $P(s_2, \phi(y^1)), P(s_2, \phi(y^2))$ and $\langle s_2, \phi(x^1), \overset{P(p(\phi(x^1)), t_1^1)}{\longrightarrow} t_1^1\rangle$. Applying Theorem 1, we can construct a k_1^*-fan $A(s_1 \to \{x^i|1 \le i \le k_1\})$ of $Q_{n-1}^0 - \{t_2^1, y^1, t_2^2, y^2\}$. Thus, the paths $\langle s_1, \overset{P(s_1, x^1)}{\longrightarrow} x^1, \phi(x^1), \overset{P(\phi(x^1), t_1^1)}{\longrightarrow} t_1^1\rangle, \langle s_1, \overset{P(s_1, x^i)}{\longrightarrow} x^i, \phi(x^i), t_1^i\rangle, \langle s_2, \overset{P(s_2, \phi(y^j))}{\longrightarrow} \phi(y^j), y^j, t_2^j\rangle$ form the

$(n-1-f_a)^*$-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$, for $2 \le i \le k_1, j = 1, 2$.

Suppose that $d(s_2, t_2^2) = 2$ and $d(s_2, t_2^1) \ge 2$. Let (y^1, t_2^1) be an edge of Q_{n-1}^0 for $\phi(y^1) \notin T^1$. Let $(\phi(x^i), t_1^i)$ be an edge of Q_{n-1}^1 for $x^i \notin \{s_1, t_2^1, t_2^2, y^1\}$, for $1 \le i \le k_1$. Applying Lemma 3, we can construct two spanning disjoint paths $P(s_2, \phi(y^1))$ and $P(\phi(x^1), t_1^1)$ of $Q_{n-1}^1 - F_a^1 - \{t_1^i, \phi(x^i)|2 \le i \le k_1\}$. Applying Theorem 1, we can also construct a k_1^*-fan $A(s_1 \to \{x^i|1 \le x \le k_1\})$ of $Q_{n-1}^0 - \{t_2^1, y^1, t_2^2, \phi(s_2)\}$. Thus, the paths $\langle s_1 \overset{P(s_1,x^1)}{\longrightarrow} x^1, \phi(x^1), \overset{P(\phi(x^1),t_1^1)}{\longrightarrow} t_1^1\rangle, \langle s_1 \overset{P(s_1,x^i)}{\longrightarrow} x^i, \phi(x^i), t_1^i\rangle, \langle s_2, \overset{P(s_2,\phi(y^1))}{\longrightarrow} \phi(y^1), y^1, t_2^1\rangle, \langle s_2, \phi(s_2), t_2^2\rangle$ form the (n-1-f_a)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$ for $2 \le i \le k_1$.

Suppose that $d(s_2, t_2^2) = 2$ and $d(s_2, t_2^1) = 1$. Let $(z, t_2^2), (y, z)$ be edges of Q_{n-1}^0 for $\{y, z, \phi(y)\} \notin (T_1 \cup T_2 \cup F_a \cup \{s_1, s_2\})$. Let $(\phi(x^i), t_1^i)$ be edges of Q_{n-1}^1 for $\phi(x^i), x^i \notin (T_1 \cup T_2 \cup F_a \cup \{s_1, s_2, y, z, \phi(y), \phi(z)\})$ and $1 \le i \le k_1$. Applying Lemma 3, we can construct two spanning disjoint paths $P(s_2, \phi(y))$ and $P(\phi(x^1), t_1^1)$ of $Q_{n-1}^1 - F_a^1 - \{t_1^i, \phi(x^i)|2 \le i \le k_1\}$. Applying Theorem 1, we can construct a k_1^*-fan $A(s_1 \to \{x^i|1 \le i \le k_1\})$ of $Q_{n-1}^0 - \{t_2^1, t_2^2, y, z\}$. Thus, the paths $\langle s_1 \overset{P(s_1,x^1)}{\longrightarrow} x^1, \phi(x^1) \overset{P(\phi(x^1),t_1^1)}{\longrightarrow} t_1^1\rangle, \langle s_1 \overset{P(s_1,x^i)}{\longrightarrow} x^i, \phi(x^i), t_1^i\rangle, \langle s_2, t_2^1\rangle, \langle s_2 \overset{P(s_2,\phi(y))}{\longrightarrow} \phi(y), y, z, t_2^2\rangle$ form the (n-1-f_a)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$.

The proof of the case $\{t_2^1, t_2^2\} \not\subset V^0$ is similar to the proof of $t_2^1, t_2^2 \in V^0$.

Case 2.2.2 $|T_2| = 1$.

Suppose that $|T_1^0| = 1$ and $|T_2^0| = 1$. Without loss of generality, we can assume that $t_1^1 \in V^0$. Let $(y, t_2^1) \in Q_{n-1}^0$ for $y, \phi(y) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\})$. Let $(\phi(x^i), t_1^i)$ be edges of Q_{n-1}^1 for $x^i, \phi(x^i) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2, y, \phi(y)\})$ for $2 \le i \le k_1$. Applying Lemma 3, we can construct two spanning disjoint paths $P(s_2, \phi(y))$ and $P(\phi(x^2), t_1^2)$of $Q_{n-1}^1 - F_a^1 - \{\phi(x^i), x^i|3 \le i \le k_1\}$. Applying Theorem 1, we can construct a k_1^*-fan $A(s_1 \to \{t_1^1, x^2, x^3, \cdots x^{k_1}\}$ of $Q_{n-1}^0 - \{t_2^1, y\}$. Thus, the paths $P(s_1, t_1^1), \langle s_1 \overset{P(s_1,x^i)}{\longrightarrow} x^i, \phi(x^i), t_1^i\rangle$, and $\langle s_2 \overset{P(s_2,\phi(y))}{\longrightarrow} \phi(y), y, t_2^1\rangle$ form the (n-1-f_a)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$.

The proofs of the cases $|T_2^0| = 0$ or $|T_1^0| = 0$ are similar to the proof of $|T_1^0| = 1$ and $|T_2^0| = 1$

Case 2.3 $|T_1^1| + |F_a^1| = n - 2$.

Suppose that $|T_2^1| = 0$. Let (y, t_2^1) be an edge of Q_{n-1}^0 for $y, \phi(y) \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2\})$. Let $(\phi(x^i), t_1^i)$ be edges of Q_{n-1}^1 for $\phi(x^i), x^i \notin (F_a \cup T_1 \cup T_2 \cup \{s_1, s_2, y, \phi(y)\})$ for $3 \le i \le k_1$. Applying Lemma 1, we can construct a Hamiltonian path $P(t_1^2, t_1^1)$ of $Q_{n-1}^1 - F_a - \{t_1^i, \phi(x^i)|3 \le i \le k_1\} - \{(\phi(y), \phi(t_2^1))\}$. Without loss of generality, we can assume that the path $P(t_1^2, t_1^1)$ can be denoted as $\langle t_1^2 \overset{P(t_1^2,\phi(x^2))}{\longrightarrow} \phi(x^2), s_2 \overset{P(s_2,\phi(y))}{\longrightarrow} \phi(y), \phi(x^1) \overset{P(\phi(x^1),t_1^1)}{\longrightarrow} t_1^1\rangle$. Applying Theorem 1, we can construct a k_1^*-fan $A(s_1 \to \{x^i|1 \le x \le k_1\}$ of $Q_{n-1}^0 - \{t_2^1, y\}$. Thus, the paths $\langle s_1 \overset{P(s_1,x^i)}{\longrightarrow} x^i, \phi(x^i) \overset{P(\phi(x^i),t_1^i)}{\longrightarrow} t_1^i\rangle, \langle s_1 \overset{P(s_1,x^j)}{\longrightarrow} x^j, \phi(x^j), t_1^j\rangle$ and $\langle s_2 \overset{P(s_2,\phi(y))}{\longrightarrow} \phi(y), y, t_2^1\rangle$ form the (n-1-f_a)*-bifan $B(s_1 \to T_1, s_2 \to T_2)$ of $Q_n - F_a$ for $1 \le i \le 2, 3 \le j \le k_1$.

The proof for $|T_2^1| = 1$ is similar to the proof for $|T_2^1| = 0$. □

IV. Conclusions and Future Works

In this paper, we have shown that $Q_n - F_a$ is (n-f_a)*-fanable and (n-1-f_a)*-bifanable if $f_a \le n - 3$ where F_a is the set of f_a pairs of adjacently faulty vertices. The vertices fault-tolerance and edges fault-tolerance for fanability and bifanability are worth studying in the future.

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