Congestion-Free Embedding of $2(n-k)$ Spanning Trees in an Arrangement Graph

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Abstract

The arrangement graph $A_{n,k}$ is not only a generalization of star graph ($n-k=1$), but also more flexible. In this investigation, we elucidate the problem of embedding of multiple spanning trees in an arrangement graph with the objective of congestion-free. This result is to report how to exploit $2(n-k)$ edge disjoint spanning trees in an arrangement graph, where each congestion-free spanning tree's height is $2k-1$. Our scheme is based on a subgraph-partitioning scheme. First, we construct $2(n-k)$ base spanning trees in every $A_{n-k+2,2}$. Then, we recursively contruct $2(n-k)$ spanning trees from every $A_{n-k+2,2}$ up to $A_{n,k}$ by a bottm-up approach. This is a near-optimal result since all of possible edges in the base subrrangement $A_{n-k+2,2}$ are fully utilized.

Keywords: Arrangement graph, interconnection network, parallel processing, spanning tree.

1. Introduction

Designing large multi-processor systems frequently involves organizing into various configurations. One of the widely studied interconnection network topologies is the star graph [9] [10] [11]. As a member of the Cayley graphs, the star graph possesses several attractive features such as its diameter-to-node-degree ratio, scalability, partitionability, symmetry, and high degree of fault tolerance [1][4]. However, the star graph is limited with respect to its number of nodes: $n!$ for an n-dimensional star graph.

A new interconnection topology, *arrangement graph,* has been proposed [5]. As a family of undirected graphs that contains the star graph family, the arrangement graph has desired properties, such as symmetric vertex and symmetric edge, strongly resilience and maximally fault-tolerance. Arrangement graph is more flexible than the star graph in terms of choosing major design parameters, i.e. member of vertices, degree and diameter, while preserving most of the excellent properties of the star graph.

The arrangement graph has received considerable attention [2][3][5][6][7][8]. Firstly, Day and Tripathi [6] designed a shortest-path routing algorithm for the arrangement graphs. According to their results, the arrangement graph can be embedded cycle whose length ranging from three to the size of the graph [5]. Moreover, the arrangement graph can be decomposed into vertex disjoint cycles in many different ways [5]. Furthermore, multi-dimensional grads, hypercube and one spanning tree can be embedded in arrangement graphs [7]. The spanning tree can support broadcasting communication in the arrangement graph. Heieh and Chen [8] further demonstrated that the arrangement graph remains a ring even if it is faulty. Bat *et al.* [2] recently propose a distributed fault-tolerant algorithm for one-to-all broadcasting only in the one-port communication model on the arrangement graph.

In light of above discussion, this work elucidates the problem of construction of multiple spanning trees in arrangement graphs. In this investigation, we assume that a node consists of a processor with bidirectional communication links to each of its adjacent nodes. Therefore, the term *edge-disjoint spanning trees* can be interchangeably adopted to reflect that no two edges of our spanning trees share a same direct communication link. To our knowledge, this work reports to the feasibility of embedding $2(n-k)$ spanning trees in an $A_{n,k}$ while, at the same time, keeping the edge-congestion free. Similar results for the star graph can be found in [12][3].

A tree is a common structure to represent inter-task communication pattern of a parallel algorithm. In this work, we propose a new spanning tree in an arrangement graph $A_{n,k}$ that has nice property that $2(n-k)$ copies of such trees can be embedded simultaneously in the network with edge-congestion free. In this paper, we consider the embedding of multiple spanning trees in an arrangement graph with the objective of

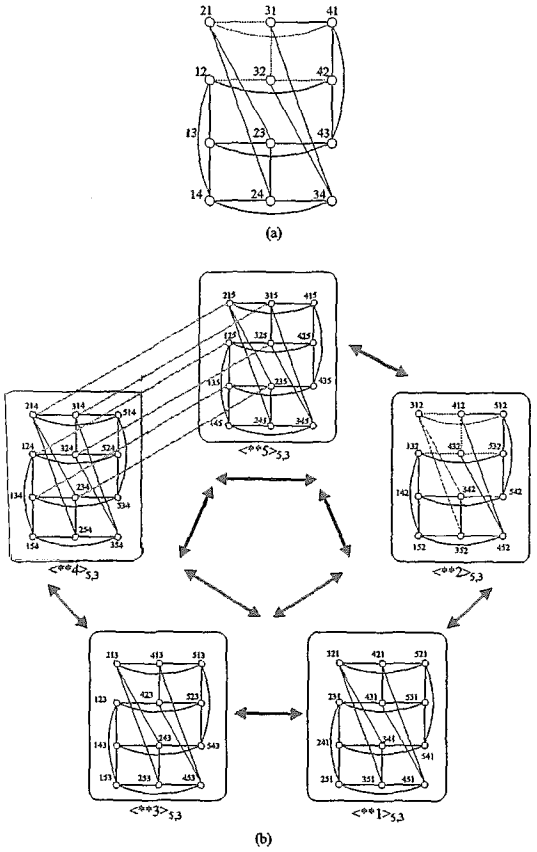

Figure 1:
The example of arrangement graph (a) $A_{4,2}$ and (b) $A_{5,3}$.

congestion free. Chen *et al.* recently proposed a scheme to embed $n-k$ spanning trees in [3]. Further, this paper exploits the double number of spanning trees than Chen *et al.*'s scheme.

The rest of this paper is organized as follows. Section 2 introduces preliminaries. Section 3 presents the scheme of embedding $2(n-k)$ spanning trees. Conclusions are finally drawn in section 4.

2. Preliminaries

The arrangement graph is denoted by $A_{n,k}$, where specified by integers n and k and $1 \leq k \leq n-1$. Denote $\langle n \rangle = \{1,2,...,n\}$. Let $P\binom{n}{k}$ be the set of permutations of k symbols taken from $\langle n \rangle$. These k symbols are denoted as $X = x_1 x_2 \cdots x_k$. Refer x_i as the i'th element of X. The *(n,k)-arrangement graph*, denoted as $A_{n,k}$, defined in [6] is an undirected graph (V,E) as follows.

$$\begin{cases} V = \{X = x_1 x_2 \cdots x_k | x_i \text{ in } \langle n \rangle \text{ and } x_i \neq x_j \text{ for } i \neq j\} \\ \qquad = P\binom{n}{k}\}, \\ E = \{(x,y) | x \text{ and } y \text{ in } V \text{ and for some } i \text{ in } \langle k \rangle, x_i \\ \qquad \neq y_j \text{ and } x_j = y_j \text{ for } j \neq i\}. \end{cases}$$

Fig. 1 depicts an example of $A_{4,2}$ and $A_{5,3}$. The edge of $A_{n,k}$ connecting neighboring nodes which differ in exactly one of their k positions. The vertices of $A_{n,k}$ are the arrangements of k elements of $\langle n \rangle$. For example, in $A_{4,2}$, the node p=41 is connected to the nodes = 42, 43, 21 and 31. An edge of $A_{n,k}$ connecting two arrangements p and q which differ only in position i, is called an *i-edge*. For all values of n and k, $A_{n,k}$ is a regular graph on $\frac{n!}{(n-k)!}$ nodes that is regular of degree $k(n-k)$, and a diameter $\lfloor \frac{3}{2}k \rfloor$ [6]. For an arrangement $X = x_1 x_2 ... x_k$, we define $EXT(X) = \langle n \rangle - \{x_1, x_2, ..., x_k\}$ to be the $n-k$ elements of $\langle n \rangle$ not appearing in the arrangement X. Let $INT(X) = \{x_1, x_2, ..., x_k\} = \langle n \rangle - EXT(x)$ to be the k elements of $<n>$ appearing in the arrangement X. For example, we consider the node p=412 in the arrangement graph $A_{5,3}$, so $EXT(p)$= {3, 5} and $INT(p)$= {1, 2, 4}.

In an $A_{n,k}$, each node p performs an adjacent function $ADJ_{x,y}(p)$ to arrive at adjacent node q, where x is the position of label of node and y is the changed label in $EXT(p)$. Given $X = x_1 \cdots x_x \cdots x_k \in INT(X)$, the adjacent function $ADJ_{x,y}(X)$ is the adjacent nodes of X obtained by changing x_x in X as y, where $1 \leq x \leq k$ and $y \in EXT(X)$. Consider an arrangement graph $A_{5,3}$, adjacent nodes for node p =412 are 413, 415, 432, 452, 312 and 512, where $EXT(p)$ = {3, 5}, $INT(p)$ ={1, 2, 4}, $ADJ_{1,3}(p) = 312$, $ADJ_{1,5}(p) = 512$, $ADJ_{2,3}(p) = 432$, $ADJ_{2,5}(p) = 452$, $ADJ_{3,3}(p) = 413$, and $ADJ_{3,5}(p) = 415$.

The $A_{n,k}$ is with recursive structure [6]. That is, an $A_{n,k}$ can be partitioned into n copies of $A_{n-1,k-1}$, each embedded $A_{n-1,k-1}$ is conveniently denoted by $<*^{k-1}\alpha>_{n,k}$, where $\alpha \in \{1,2,\cdots,n\}$, (where $*$ represents a "don't care" symbol). For example, $<**3>_{4,3}$ represents an embedded $A_{3,2}$ of $A_{4,3}$, contains six nodes: 123, 143, 213, 243, 413, and 423. From other point of view, there are n copies of $<*^{k-1}\alpha>_{n,k}$ which are obtained by performing a split operation on $<*^{k}>_{n,k}$, where $\alpha \in \{1,2,...,n\}$. This operation is called k-partition. Generally, $A_{n,k}$ can be partitioned into $\frac{n!}{(n-p)!}$ node-disjoint copies of $A_{n-p,k-p}$ in $\frac{n!}{p!(n-p)!}$ different ways and that in total $A_{n,k}$ contains $\binom{k}{p}\frac{n!}{(n-p)!}$ copies of $A_{n-p,k-p}$, for $1 \leq p \leq k-1$.

3. Congestion-Free Embedding of $2(n-k)$ Spanning Trees

This section presents a novel embedding scheme of embedding $2(n-k)$ edge-disjoint spanning trees.

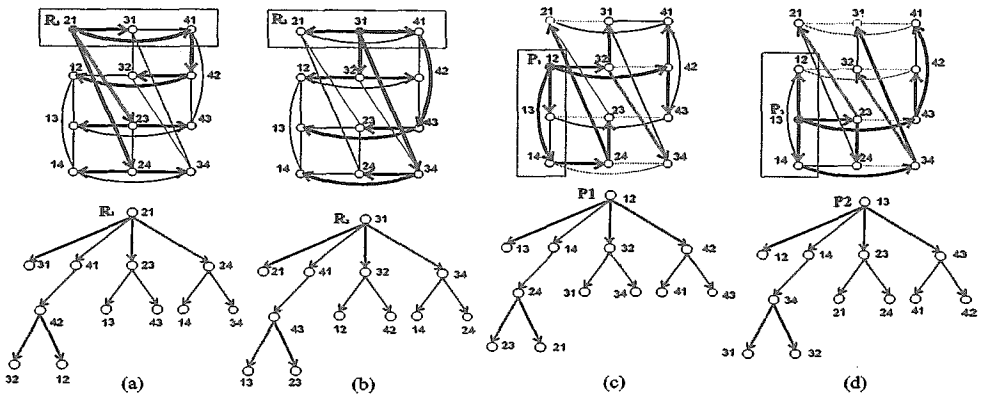

Figure 2: Four base edge-disjoint spanning trees in an $A_{4,2}$.

Our construction scheme adopts a bottom-up manner. An $A_{n,k}$ can be partitioned into $\frac{n!}{(n-k+2)!}$ copies of $A_{n-k+2,2}$. Each $A_{n-k+2,2}$ initially construct $2(n-k)$ base spanning trees. For each $A_{n-k+3,3}$, there are $n-k+3$ copies of $A_{n-k+2,2}$. Each $A_{n-k+3,3}$ will perform a concatenation operation among $n-k+3$ copies of $A_{n-k+2,2}$ to construct $2(n-k)$ spanning trees in the $A_{n-k+3,3}$. Recursively performing the concatenation operations allow us to finally construct $2(n-k)$ spanning trees in an $A_{n,k}$.

3.1. Phase 1: Generate $2(n-k)$ Base Spanning Trees in an $A_{n-k+2,2}$

After a spliting-operation on $A_{n,k}$, $\frac{n!}{(n-k+2)!}$ copies of $A_{n-k+2,2}$'s are obtained. According to different values of n and k, $2(n-k)$ base spanning trees in each $A_{n-k+2,2}$ can be generated as follows.

- Step 1: Locate $2(n-k)$ roots nodes.
- Step 2: Generate $2(n-k)$ base spanning trees in each $A_{n-k+2,2}$.

3.1.1. Step 1: Locating $2(n-k)$ Roots in an $A_{n-k+2,2}$

This section describes how to locate $2(n-k)$ root nodes in an $A_{n-k+2,2}$. An $A_{n-k+2,2}$ is partitioned into $n-k+2$ copies of $A_{n-k+1,1}$, where each $A_{n-k+1,1}$ is a $(n-k+1)$-node complete graph. The $SWP_{1,2}(R)$ function is defined to swap the first and second bits of R. Denote these $2(n-k)$ root nodes as $R_1, R_2, ..., R_{n-k}$ and $P_1, P_2, ..., P_{n-k}$, which are constructed as follows.

- Let $R_1 = (y_1x_2x_3\cdots x_k)$ be any node of one $A_{n-k+1,1}$. Other root nodes $R_2 = (y_2x_2x_3\cdots x_k)$, $R_3 = (y_3x_2x_3\cdots x_k)$, ..., and $R_{n-k+1} = (y_{n-k+1}x_2x_3\cdots x_k)$ by exchanging the first bit of R_1 with α, where $\alpha \in EXT(R_1) = \{y_2, ..., y_{n-k+1}\}$. This work is achieved as follows.

  $$R_i = ADJ_{1,\alpha}(R_1), \text{ where } 2 \le i \le n-k+1 \text{ and } \alpha \in EXT(R_1).$$

  Clearly, each pair of $R_1, R_2, ...,$ and R_{n-k+1} are adjacent since $R_1, R_2, ...,$ and R_{n-k+1} belong to an $A_{n-k+1,1}$ (a complete graph $(*x_2x_3\cdots x_k)$). Note that node R_{n-k+1} is used to be a template node to ensure congestion-free in our embedding.

- Let $P_i = SWP_{1,2}(R_i)$, where $1 \le i \le n-k+1$. Note that $P_1, P_2, ...,$and P_{n-k+1} also belong to a distinct $A_{n-k+1,1}$ (a complete graph $(x_2 * x_3\cdots x_k)$). Similarly node P_{n-k+1} is used to be a template node to ensure congestion-free in our embedding.

Intuitively, every R_i is only different in the first bit, and every P_i is only different in the second bit. Root nodes $R_i, 1 \le i \le n-k$, are selected from $(*x_2x_3\cdots x_k)$. Other root nodes $P_i, 1 \le i \le n-k$, are selected from $(x_2 * x_3\cdots x_k)$. Clearly, both $(*x_2x_3\cdots x_k)$ and $(x_2 * x_3\cdots x_k)$ are $A_{n-k+1,1}$ or $(n-k+1)$-node complete graph. Note that root nodes $R_1, R_2, ..., R_{n-k}$ and $P_1, P_2, ..., P_{n-k}$ satisfy the following property.

> **Root-Location Property:** There are $2(n-k)$ root nodes R_i and P_i, $1 \le i \le n-k$, in same subarrangement graph. Root nodes R_i, $i = 1..n-k$, belong to an $A_{n-k+1,1}$, P_i, $i = 1..n-k$, belong to a distinct $A_{n-k+1,1}$, so $P_i = SWP_{1,2}(R_i)$, for $i = 1..n-k$.

Root nodes R_i and P_i, $i = 1..n-k$, are used to expand $2(n-k)$ base spanning trees in an $A_{n-k+2,2}$. Fig. 2 illustrates an example by setting $R_1 = 21$, $R_2 = 31$ and $P_1 = 12$, $P_2 = 13$. Node $R_3 = 41$ and $P_3 = 14$ are the template nodes. This root-location property is very useful during constructing $2(n-k)$ spanning trees.

3.1.1. Step 2: Generate $2(n-k)$ Base Edge-Disjoint Spanning Trees in an $A_{n-k+2,2}$

The $2(n-k)$ base spanning trees are constructed from root R_i and P_i, $1 \le i \le n-k$, in an $A_{n-k+2,2}$. The fact that each $A_{n-k+1,1}$ is a complete graph accounts for why $2(n-k)$ base spanning trees can be constructed if we can connect each root nodes to distinct node of all other $A_{n-k+1,1}$'s. This is because that we have the following result and Fig. 3 gives an example.

Lemma 1 *For a κ-node complete graph G, there exist κ disjoint spanning trees with height one. Notably, each node in G is the root node of each spanning tree.*

Now we explain how to construct $2(n-k)$ base spanning trees in each $A_{n-k+2,2}$ such that each tree's height is 3. Given $2(n-k)$ root nodes R_i and P_i, $1 \le i \le n-k$. Consider any pair of nodes $R \in R_i$ and $P \in P_i$, where $R = SWP_{1,2}(P)$. There are $n-k$ pairs. As stated in Section 2, a n-dimensional arrangement graph contains n subarrangements that we use to derive the desired spanning trees. Let R and P are one of these; we must connect R and P to other $n-1$ subarrangements. For each of R and P, we use a single edge to connect the R and P to $n-k$ of subarrangements. We use an intermediate node in same subarrangement as the bridge node to connect R and P to the remaining $k-1$ subarrangements by two edges. If $k=2$, then there is only one subarrangement connecting by two edges.

Rules **A1** and **A2** formalize our basic spanning tree construction.

A1: (Single edge) Let nodes $\dot{R}$ and $\dot{P}$ denote two nodes in every $n-k$ subarrangements.

$$\begin{cases} \dot{R} = ADJ_{2,\alpha}(R), \text{ where } \alpha \in EXT(R). \\ \dot{P} = ADJ_{2,\alpha}(P), \text{ where } \alpha \in EXT(P). \end{cases}$$

Note that $\dot{P} = SWP_{1,2}(\dot{R})$.

A2: (Two edges) Let nodes $\ddot{R}$ and $\ddot{P}$ denote as two nodes in one of remaining $k-1$ subarrangements which connected by two edges (Note that in this case, $k=2$). Recall in phase 1, nodes R_{n-k+1} and P_{n-k+1} are the intermediate nodes for nodes R and P, respectively. Note that $R_{n-k+1} = SWP_{1,2}(P_{n-k+1})$.

$$\begin{cases} \ddot{R} = ADJ_{2,\beta}(R_{n-k+1}), \text{where } \beta \text{ is the first bit value in } R. \\ \ddot{P} = ADJ_{2,\beta}(P_{n-k+1}), \text{where } \beta \text{ is the second bit value in } P. \end{cases}$$

Note that $\ddot{P} = SWP_{1,2}(\ddot{R})$. Fig. 2(a) shows that root node 21 uses distinct single edge to connecting nodes 23 and 24, and connecting node 42 by two edges. Fig. 2(b) displays that root node 31 directly connects to 32 and 34 but connects 43 by two edges. Fig. 2(c) gives example for other root nodes 12 and 13.

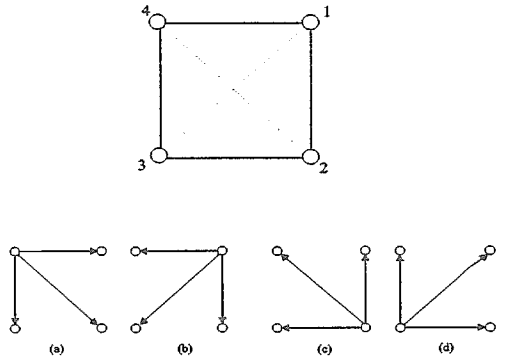

Figure 3:
Four spanning trees with height 1 in a 4-node complete graph.

In the following, we describe the existence of $2(n-k)$ base spanning trees. Some notations are defined firstly. Given root nodes R_i and P_i, $1 \le i \le n-k$, are constructed by phase 1. An $A_{n-k+2,2}$ can be partitioned into $n-k+2$ copies of $A_{n-k+1,1}$ or $A'_{n-k+1,1}$ along dimension two or one. First, assume that an $A_{n-k+2,2}$ is partitioned into $n-k+2$ copies of $A_{n-k+1,1}$ along second dimension, where R_i, $1 \le i \le n-k$, located in one of $A_{n-k+1,1}$. Let $IE(R_i)$, $1 \le i \le n-k$, denote a set of all possible internal edges within each $A_{n-k+1,1}$. Let $EE(R_i)$ denote a set of all possible external edges outgoing each $A_{n-k+1,1}$. Secondly, an $A_{n-k+2,2}$ is partitioned into $n-k+2$ copies of $A'_{n-k+1,1}$ along first dimension, where P_i, $1 \le i \le n-k$, located in one of $A'_{n-k+1,1}$. Let $IE(P_i)$, $1 \le i \le n-k$, denote a set of all possible internal edges within each $A'_{n-k+1,1}$. Let $EE(P_i)$ denote a set of all possible external edges outgoing each $A'_{n-k+1,1}$. Further, let $E(x,y)$ denote as edge beginning from node x to destination node y. Note that $E(x,y)$ and $E(y,x)$ represent different edges since the link is assumed as full duplex. For example, as illustrated in Fig. 4(a), all bold edges are $IE(R_i)$ and all dash edges are $EE(R_i)$. In Fig. 4(b), all bold edges are $IE(P_i)$ and all dash edges are $EE(P_i)$.

Some important properties are used later as stated herein. For $1 \le i \le n-k$, we have following properties.

P1: Edges in $IE(R_i)$ are equal to edges in $EE(P_i)$.

Fig. 4(a) shows that $E(21,31) \in IE(R_i)$ and $E(21,31) \in EE(P_i)$.

P2: Edges in $IE(P_i)$ are equal to edges in $EE(R_i)$.

Fig. 4(b) displays that $E(21,23) \in IE(P_i)$ and $E(21,23) \in EE(R_i)$.

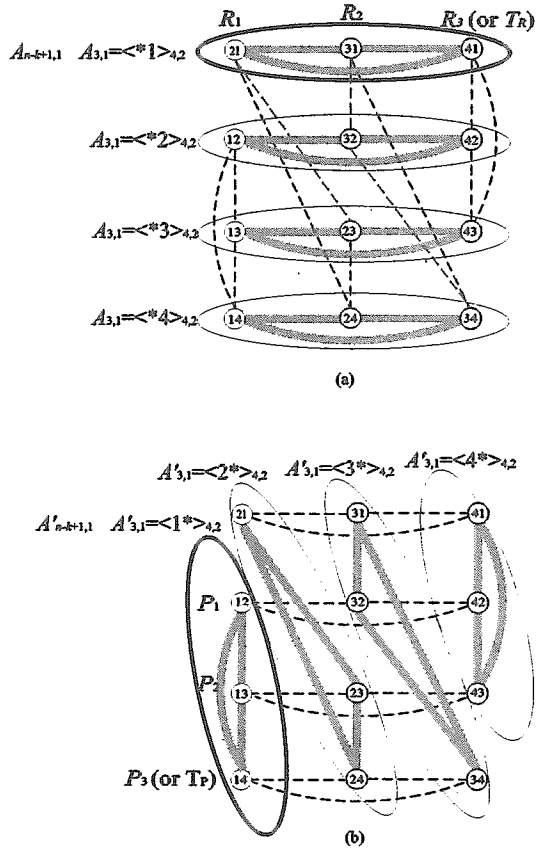

Figure 4: (a) All internal edges , (b) all internal edges.

P3: Edges in $IE(R_i)$ and edges in $EE(R_i)$ are equal to all edges in the $A_{n-k+2,2}$.

P4: Edges in $IE(R_i)$ and edges in $EE(R_i)$ are distinct.

P5: Edges in $IE(P_i)$ and edges in $EE(P_i)$ are distinct.

Lemma 2 *There exist $n-k$ edge-disjoint spanning trees $ST_{n-k+2,2}(R_i)$ in an $A_{n-k+2,2}$, for $1 \leq i \leq n-k$.*

Proof. Given root nodes R_i, $1 \leq i \leq n-k$, are located in one of partitioned $A_{n-k+1,1}$. Every root node needs to connect to other $A_{n-k+1,1}$. Therefore, $n-k$ spanning trees $ST_{n-k+2,2}(R_i)$, for $1 \leq i \leq n-k$, are mutually disjoint due to the fact that every root node satisfy the following conditions:

1: All edges of R_1, R_2,..., and R_{n-k} connecting to the same $A_{n-k+1,1}$ are disjoint.

2: All nodes in the same $A_{n-k+1,1}$ connecting to R_1, $R_2, \ldots$, and R_{n-k} differ from each other.

The reason is stated as follows. Recalled again, since $k=2$, for all R_i, we use $n-2$ distinct single edges connecting to $n-2$ copies of $A_{n-k+1,1}$ and use two edge to connect with remaining one $A_{n-k+1,1}$. Intuitively, all edges connecting from R_1, R_2,..., and R_{n-k} to template node R_{n-k+1} are distinct since all of these nodes located in a $(n-k+1)$-node complete graph (or $A_{n-k+1,1}$). Recall previous notation, edges in all possible $A_{n-k+1,1}$ are denoted as $IE(R_i)$ and all edges in all possible $A'_{n-k+1,1}$ are denoted as $EE(R_i)$. For condition 1, all edges of R_1, R_2,..., and R_{n-k} connecting to the same $A_{n-k+1,1}$ are disjoint because every edge belongs to different $A'_{n-k+1,1}$. Remember, these edges are belong to $EE(R_i)$. For condition 2, all nodes in the same $A_{n-k+1,1}$ connecting to $R_1, R_2, \ldots$, and R_{n-k} differ from each other due to the fact that each of these nodes belongs to distinct $A'_{n-k+1,1}$. ■

Based on node symmetry, we have the following similar result.

Lemma 3 *There exist $n-k$ edge-disjoint spanning trees $ST_{n-k+2,2}(P_i)$ in an $A_{n-k+2,2}$, for $1 \leq i \leq n-k$.*

To make the clear description of lemma 6, some notations are defined. Let $SE(R_i,A)$, $A \in A_{n-k+1,1}$ or $A'_{n-k+1,1}$, denote a set of spanning edges which exist in tree $ST(R_i)$ belonging to subarrangement A. In the same way, $SE(P_i,A)$,$A \in A_{n-k+1,1}$ or $A'_{n-k+1,1}$, denote a set of spanning edges which exist in tree $ST(P_i)$ belonging to subarrangement A. For example in Fig. 4(b), edges $E(21,23)$ and $E(21,24)$ are belong to $SE(21,<2*>) \in EE(21)$. Edges $E(24,21)$ and $E(24,23)$ are belong to $SE(12,<2*>) \in IE(12)$.

Two other important properties are stated. For $1 \leq i \leq n-k$, we have the following properties.

P6:
$$\begin{cases} \text{Spanning edges in tree } ST(R_i) \text{ belonging to} \\ \quad A_{n-k+1,1}, \text{ i.e., } SE(R_i, A_{n-k+1,1}), \in IE(R_i). \\ \text{Spanning edges in tree } ST(R_i) \text{ belonging to} \\ \quad A'_{n-k+1,1}, \text{ i.e., } SE(R_i, A'_{n-k+1,1}), \in EE(R_i). \end{cases}$$

For instance as shown in Figs. 4(a) and 4(b), $E(24,14) \in SE(21,<*4>_{4,2}) \in IE(21)$ and $E(21,24) \in SE(21,<2*>_{4,2}) \in EE(21)$.

P7:
$$\begin{cases} \text{Spanning edges in tree } ST(P_i) \text{ belonging to} \\ \quad A'_{n-k+1,1}, \text{i.e., } SE(P_i, A'_{n-k+1,1}), \in IE(P_i). \\ \text{Spanning edges in tree } ST(P_i) \text{ belonging to} \\ \quad A_{n-k+1,1}, \text{i.e., } SE(P_i, A_{n-k+1,1}), \in EE(P_i). \end{cases}$$

For instance as illustrated in Figs. 4(b) and 4(a), $E(24,21) \in SE(12,<2*>_{4,2}) \in IE(12)$ and $E(14,24) \in SE(12,<*4>_{4,2}) \in EE(12)$.

Lemma 4 *Each pair of $ST_{n-k+2,2}(R_i)$ and $ST_{n-k+2,2}(P_i)$, for $1 \leq i \leq n-k$, are mutually disjoint.*

Proof. Initially, all edges of root nodes R_i connecting to template node R_{n-k+1} located in one $(n-k+1)$-node complete graph, namely $\hat{A}_{n-k+1,1}$, and all edges of root nodes P_i connecting to template node P_{n-k+1}

located in another one $(n-k+1)$-node complete graph, namely $\hat{A}'_{n-k+1,1}$. Note that, all spanning trees $ST_{n-k+2,2}(R_i)$ do not use any internal edge of $\hat{A}'_{n-k+1,1}$ and all spanning trees $ST_{n-k+2,2}(P_i)$ do not use any internal edge of $\hat{A}_{n-k+1,1}$. For example as shown in Fig. 4, edges for R_1 and R_2 connecting to template node R_{n-k+1} belong to a complete graph $A_{n-k+1,1} = A_{3,1} =< *1 >_{4,2}$. And edges for P_1 and P_2 connecting to template node P_{n-k+1} belong to another complete graph $A'_{n-k+1,1} = A'_{3,1} =< 1* >_{4,2}$. Clearly, all of these edges are disjoint. Based on $\mathbb{P}1$ and $\mathbb{P}2$ property, for all $ST_{n-k+2,2}(R_i)$ and $ST_{n-k+2,2}(P_i), 1 \le i \le n-k$, may spread its subtree in same subarrangement $A'_{n-k+1,1}$ or $A_{n-k+1,1}$. For an $A_{n-k+1,1}$ containing root node R_i, $1 \le i \le n-k$, using $n-1$ edges connecting to other $A_{n-k+1,1}$ denote as $SE(R_i, A'_{n-k+1,1}) \in EE(R_i) = IE(P_i)$. Each pair of $ST_{n-k+2,2}(R_i)$ and $ST_{n-k+2,2}(P_i)$, for $1 \le i \le n-k$, are said to be mutually disjoint if and only if we can prove that $SE(R_i, A'_{n-k+1,1})$ and $SE(P_i, A'_{n-k+1,1}), 1 \le i \le n-k$ are edge-disjoint and $SE(R_i, A_{n-k+1,1})$ and $SE(P_i, A_{n-k+1,1}), 1 \le i \le n-k$, are edge-disjoint. These two cases are stated as follows.

1: Initially, we show that all of spanning tree edges in tree $ST(R_i)$ and $ST(P_i)$ belonging to same subarrangement $A'_{n-k+1,1}$, i.e., $SE(R_i, A'_{n-k+1,1})$ and $SE(P_i, A'_{n-k+1,1}), 1 \le i \le n-k$, are edge-disjoint. Note that, $SE(R_i, A'_{n-k+1,1}) \in EE(R_i) = IE(P_i)$ and $SE(P_i, A'_{n-k+1,1}) \in IE(P_i) = EE(R_i)$. The reason of $SE(R_i, A'_{n-k+1,1})$ and $SE(P_i, A'_{n-k+1,1})$ being edge-disjoint is stated as follows. For every spanning tree $ST_{n-k+2,2}(P_i)$ may spread its subtree in same $A'_{n-k+1,1}$ beginning from a node, namely first node. $SE(R_i, A'_{n-k+1,1})$ and $SE(P_i, A'_{n-k+1,1})$ are edge-disjoint if all of first nodes are different in same $A'_{n-k+1,1}$. In every same $A'_{n-k+1,1}$, $SE(R_i, A'_{n-k+1,1}) \in EE(R_i) = IE(P_i)$ and $SE(P_i, A'_{n-k+1,1}) \in IE(P_i) = EE(R_i), 1 \le i \le n-k$, are edge-disjoint because the corresponding first node is different. This is because that all first nodes are located in different $A_{n-k+1,1}$.

2: Next, we prove that all of spanning tree edges in tree $ST(R_i)$ and $ST(P_i)$ belonging to same subarrangement $A_{n-k+1,1}$, i.e, $SE(R_i, A_{n-k+1,1})$ and $SE(P_i, A_{n-k+1,1}), 1 \le i \le n-k$, are edge-disjoint. Note that $SE(R_i, A_{n-k+1,1}) \in IE(R_i) = EE(P_i)$ and $SE(P_i, A_{n-k+1,1}) \in EE(P_i) = IE(R_i)$. Due to node symmetry, so we omit the detail, but $SE(R_i, A_{n-k+1,1})$ and $SE(P_i, A_{n-k+1,1}), 1 \le i \le n-k$ are edge-disjoint in the same $A_{n-k+1,1}$ by the similar reason in the case 1. ■

Fig. 4 illustrates an example that let $R_1 = 21$, it has two direct edges $SE(21, <2*>_{4,2}) \in EE(21)$ connecting to subarrangement $<*3>_{4,2}, <*4>_{4,2} = A_{3,1}$ with nodes 23,24 and one edge $SE(21, <4*>_{4,2}) \in EE(21)$ for node 41 connecting to another subarrangement $<*2>_{4,2} = A_{3,1}$ with node 42. Spanning tree $ST_{4,2}(R_1), ST_{4,2}(P_1)$ and $ST_{4,2}(P_2)$ respectively spread their subtrees in $<2*>_{4,2} = A'_{3,1}$ by first node 21,24,23 and spread subtrees in $<4*>_{4,2} = A'_{3,1}$ by first node 41,42,43. We can find that the edges for all spanning trees spread its subtrees in subarrangement $<2*>_{4,2} = A'_{3,1}$ and $<4*>_{4,2} = A'_{3,1}$ are disjoint from each other due to all the first nodes are different and locate in different $A_{3,1}$.

Theorem 1 *There exist $2(n-k)$ base edge-disjoint spanning trees $ST_{n-k+2,2}(R_i)$ and $ST_{n-k+2,2}(P_i)$ in an $A_{n-k+2,2}$, where $1 \le i \le n-k$. Each spanning tree's height is 3.*

Proof. Given root nodes R_i and P_i, where $1 \le i \le n-k$. First, there exist $n-k$ spanning trees $ST_{n-k+2,2}(R_i)$ and $ST_{n-k+2,2}(P_i)$ in an $A_{n-k+2,2}$, for $1 \le i \le n-k$, in Lemmas 2 and 3. Lemma 4 illustrates that each pair of $ST_{n-k+2,2}(R_i)$ and $ST_{n-k+2,2}(P_i)$, for $1 \le i \le n-k$, are mutually disjoint. Hence the Theorem. ■

3.2. Phase 2: Construction of $2(n-k)$ Edge-Disjoint Spanning Trees in an $A_{n,k}$

The above section constructed $2(n-k)$ base spanning trees in $A_{n-k+2,2}$, and each base spanning tree's height is 3. In this section, we describe how to construct $2(n-k)$ spanning trees $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i)$, for $1 \le i \le n-k$ in an $A_{n,k}$, where the height of each spanning tree is $2k-1$.

For induction, we construct $2(n-k)$ spanning trees in an $A_{n,k}$ by using $2n(n-k)$ spanning tress in n copies of $A_{n-1,k-1}$. Our major task is to connect $2n(n-k)$ spanning subtrees into $2(n-k)$ spanning trees $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i), 1 \le i \le n-k$.

Consider an $A_{n,k}$ which is partitioned into n copies of $A_{n-1,k-1}$. Assume that $2(n-k)$ spanning trees $ST_{n-1,k-1}(R_i)$ and $ST_{n-1,k-1}(P_i)$, $i = 1..n-k$, can be constructed in each $A_{n-1,k-1}$ if their root nodes satisfied the root-location property. A randomly selected $A_{n-1,k-1}$ servers as beginning spanning tree. From this $A_{n-1,k-1}$, there exist $2(n-k)$ spanning trees $ST_{n-1,k-1}(R_i)$ and $ST_{n-1,k-1}(P_i), 1 \le i \le n-k$, where R_i and P_i are root nodes. Let root nodes R_i and P_i connect to root nodes R'_i and P'_i, where R'_i and P'_i located in other $A_{n-1,k-1}$. If R'_i and P'_i satisfy the root-location property, therefore we can embedd $2(n-k)$ spanning trees $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i), 1 \le i \le n-k$, on each of the $A_{n-1,k-1}$, so we can recursively construct $2(n-k)$ spanning trees in an $A_{n,k}$.

Given $2(n-k)$ root nodes R_i and P_i, $1 \le i \le n-k$. Consider any pair of nodes $R \in R_i$ and $P \in P_i$, where

$R = SWP_{1,2}(P)$. Note that there are $n-k$ pairs. A n-dimensional arrangement graph contains n subarrangements that we use to derive the desired spanning trees. Let R and P are one of these; we must connect R and P to other $n-1$ subarrangements. For each of R and P, we use a single edge to connect the R and P to $n-k$ of subarrangements. We also use an intermediate node in same subarrangement as the bridge node to connect R and P to the remaining $k-1$ subarrangements by two edges.

Rules **A1'** and **A2'** formalize our recursively spanning tree construction.

A1': (Single edge) The part is same as **A1**. Let nodes $\dot{R}$ and $\dot{P}$ denote two nodes in every $n-k$ subarrangements.

$$\begin{cases} \dot{R} = ADJ_{2,\alpha}(R), \text{ where } \alpha \in EXT(R). \\ \dot{P} = ADJ_{2,\alpha}(P), \text{ where } \alpha \in EXT(P). \end{cases}$$

Note that $\dot{P} = SWP_{1,2}(\dot{R})$.

A2': (Two edges) This part is same as **A2**. Let nodes $\ddot{R}$ and $\ddot{P}$ denote as two nodes in one of remaining $k-1$ subarrangements which connected by two edges. Nodes R_{n-k+1} and P_{n-k+1} are the intermediate nodes for nodes R and P, respectively. Note that $R_{n-k+1} = SWP_{1,2}(P_{n-k+1})$.

$$\begin{cases} \ddot{R} = ADJ_{2,\beta}(R_{n-k+1}), \text{where } \beta \text{ is the first bit value in } R. \\ \ddot{P} = ADJ_{2,\beta}(P_{n-k+1}), \text{where } \beta \text{ is the second bit value in } P. \end{cases}$$

Note that $\ddot{P} = SWP_{1,2}(\ddot{R})$. (See example in Fig. 5).

Assume that R'_i and P'_i, $1 \le i \le n-k$, are new connecting root nodes in each of other $(n-1)$-subarrangement. Ensuring that root nodes R'_i and P'_i satisfied the root-location property would allow us to establish our spanning trees $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i)$, for $i = 1..n-k$. In the following, we show the correctness that root nodes R'_i and P'_i satisfied root-location property, for $i = 1..n-k$.

Lemma 5 *There are $2(n-k)$ root nodes R'_i and P'_i, $1 \le i \le n-k$, in same subarrangement graph are satisfied the root-location property.*

Proof. Without loss of generality, assume that $n-k$ root nodes R_i, $1 \le i \le n-k$, represented as $(aby_3y_4\cdots y_k) \in <**y_3y_4\cdots y_k>_{n,k}$ and P_i, $1 \le i \le n-k$, represented as $(bay_3y_4\cdots y_k) \in <**y_3y_4\cdots y_k>_{n,k}$, R_i and P_i connect to R'_i and P'_i, $1 \le i \le n-k$, respectively. Our results demonstrate that root nodes R'_i and P'_i, $1 \le i \le n-k$, in same subarrangement graph are satisfied the root-location property. Three possible cases are discussed.

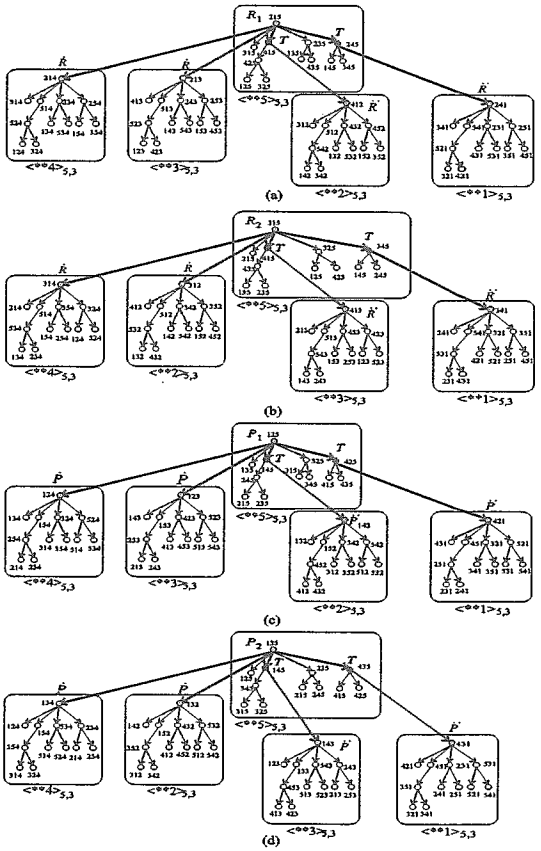

Figure 5: Example of four spanning trees in an $A_{5,3}$.

1. (Both links are single edge) Since $EXT(R_i = (aby_3y_4..y_k)) = EXT(P_i = (bay_3y_4\cdots y_k))$, so it is possible to let R'_i =$(aby_3y_4\cdots y_{k-1}\gamma)$ and P'_i =$(bay_3y_4\cdots y_{k-1}\gamma)$, where γ belongs to a set of elements which are not used in all root nodes $R_1,\ldots,$ and R_{n-k} or $P_1,...,$ and P_{n-k}, which can be represented as $\gamma \in EXT(R_1) \cap EXT(R_2) \cap \cdots \cap EXT(R_{n-k}) = EXT(P_1) \cap EXT(P_2) \cap ... \cap EXT(P_{n-k})$. Obviously, $P'_i = SWP_{1,2}(R'_i) \in <** y_3y_4\cdots y_{k-1}\gamma>_{n,k}$ is satisfied the root-location property.

2. One link is single link, other link is two edges) For $1 \le i \ne j \le n-k$, let $R_i = (\alpha by_3y_4\cdots y_k)$, $R_j = (\beta by_3y_4\cdots y_k)$, $P_i = (b\alpha y_3y_4\cdots y_k)$, and $P_j = (b\beta y_3y_4\cdots y_k) \in <**y_3y_4\cdots y_k>_{n,k}$. Note that $P_i = SWP_{1,2}(R_i)$, $P_j = SWP_{1,2}(R_j)$. First, if $\beta \in EXT(R_i) = EXT(P_i)$, then $R'_i = (\alpha by_3y_4\cdots y_{k-1}\beta)$ and $P'_i = (b\alpha y_3y_4\cdots y_{k-1}\beta)$. Second, if $\beta \in INT(R_j) = INT(P_j)$, two edges are needed for R_j and P_j to connect R'_j and P'_j in subarrangement $<** y_3y_4\cdots y_{k-1}\beta>_{n,k}$, respectively. Then let $\omega \in EXT(R_j)$, we can obtain $R_j = (\beta by_3y_4\cdots y_k) \rightarrow$

$(\omega by_3y_4\cdots y_k) \rightarrow (\omega by_3y_4\cdots y_{k-1}\ \beta) = R'_j$ and $P_j = (b\beta y_3y_4\cdots y_k) \rightarrow (b\omega y_3y_4\cdots y_k) \rightarrow (b\omega y_3y_4\cdots y_{k-1}\beta) = P'_j$. Therefore, $P'_i = SWP_{1,2}(R'_i)$ and $P'_j = SWP_{1,2}(R'_j) \in < * * y_3y_4\cdots y_{k-1}\beta >_{n,k}$ are satisfied the root-location property.

3. (Both links are two edges) For $1 \leq i \leq n-k$, let $R_i = (aby_3y_4\cdots y_k)$ and $P_i = (bay_3y_4\cdots y_k)$. Because of $b \in INT(R_i) = INT(P_i)$, two edges are needed for R_i and P_i to connect R'_i and P'_i in subarrangement $<**y_3y_4\cdots y_{k-1}b>_{n,k}$, respectively. Without loss of generality, if γ belongs to a set of elements which are not used in root nodes $R_1,\ldots,$ and R_{n-k}, we can obtain that $R_i = (aby_3y_4\cdots y_k) \rightarrow (a\gamma y_3y_4\cdots y_k) \rightarrow (a\gamma y_3y_4\cdots y_{k-1}b) = R'_i$, and $P_i = (bay_3y_4\cdots y_k) \rightarrow (\gamma ay_3y_4\cdots y_k) \rightarrow (\gamma ay_3y_4\cdots y_{k-1}b) = P'_i$. Therefore, $P'_i = SWP_{1,2}(R'_i) \in <**y_3y_4\cdots y_{k-1}b>_{n,k}$ is satisfied the root-location property. ■

Lemma 6 *All edges for R_i and $P_i, 1 \leq i \leq n-k$, respectively connect to root nodes R'_i and P'_i which located in other subarrangement graphs are all edge-disjoint.*

Proof. Note that root nodes R_i and $P_i, 1 \leq i \leq n-k$, are all distinct, and R_i and P_i use same rule to connect with R'_i and P'_i, therefore R_i and P_i are also different from each other. Because of different source and destination, all edges for R_i and P_i respectively connect to root nodes R'_i and P'_i which located in other subarrangement graphs are all edge-disjoint. ■

Lemma 7 *The height of $2(n-k)$ edge disjoint spanning trees $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i)$ in an $A_{n,k}$ is $2k-1$.*

Proof. There are k steps to construct $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i)$, $1 \leq i \leq n-k$. Each step needs at most two edges except that the final step in $A_{n-k+1,1}$ needs one single edge since $A_{n-k+1,1}$ is a complete graph. The height is $2(k-1)+1 = 2k-1$. ■

Theorem 2 *There exist $2(n-k)$ edge-disjoint spanning trees $ST_{n,k}(R_i)$ and $ST_{n,k}(P_i)$ with height $2k-1$ in an $A_{n,k}$, where $i = 1..n-k$.*

4. Conclusions

In this paper, we consider the problem of congestion-free embedding of multiple spanning trees in an arrangement graph. This is first result to exploit multiple spanning trees. In this paper, we develop a bottom-up congestion-free embedding of $2(n-k)$ spanning trees with height $2k-1$ in an (n,k)-dimensional arrangement graph. A well known application of edge-disjoint multiple spanning trees is the broadcasting algorithm. Work is currently underway to develop the optimal number of spanning trees in $A_{n,k}$ by exploiting $O(k(n-k))$ edge-disjoint spanning trees.

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References

[1] S. B. Akers, D. Harel, and B. Krishnameurthy. The star graph: An attractive alternative to the n-cube. In *Proceedings of International Conference on Parallel Processing*, pages 393–400, aug. 1987.

[2] L. Q. Bai, H. Ebara, H. Nakano, and H. Maeda. Fault-tolerant broadcasting on the arrangement graph. *The Computer Journal*, 41(3):171–184, 1998.

[3] Y.-S. Chen, T.-Y. Juang, and E.-H. Tseng. Efficient broadcast in an arrangement graph using multiple spanning trees. *To appear in IEICE Transactions on Fundamentals of Electronic, Communications, and Computer Science*, 2000.

[4] Y.-S. Chen and J.-P. Sheu. A fault-tolerant reconfiguration scheme in the faulty star graph. *Journal of Information Science and Engineering*, 16(1), Jan. 2000.

[5] K. Day and A. Tripathi. Characterization of node disjoint path in arrangement graphs. In *Technical Report TR91-43, Computer Science Department, University of Minnesota*, 1991.

[6] K. Day and A. Tripathi. Arrangement graph a: A class of generalized star graphs. *Information Processing Letter*, 42(5):235–241, 1992.

[7] K. Day and A. Tripathi. Embedding grids, hypercube, and trees in arrangement graphs. In *Proceedings of International Conference on Parallel Processing*, volume III, pages 65–72, 1993.

[8] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho. Fault-free hamiltonian cycles in faulty arrangement graphs. *IEEE Transactions on Parallel and Distributed Systems*, 10(3):223–237, mar. 1999.

[9] J.-P. Sheu, C.-T. Wu, and T.-S. Chen. An optimal broadcasting algorithm without message redundancy in star graphs. *IEEE Transactions on Parallel and Distributed Systems*, 6(6):653–658, 1995.

[10] Y.-C. Tseng, S.-H. Chang, and J.-P. Sheu. Fault-tolerant ring embedding in star graphs. *IEEE Transactions on Parallel and Distributed Systems*, 8(12):1185–1195, dec. 1997.

[11] Y.-C. Tseng, Y.-S. Chen, T.-Y. Juang, and C.-J. Chang. Congestion-free, dilation-2 embeding of complete binary tree in star graphs. *Networks*, 33(3):221–231, May 1999.

[12] Y.-C. Tseng and J.-P. Sheu. Toward optimal broadcast in a star graph using multiple spanning trees. *IEEE Transactions on Computers*, 46(5):593–599, 1997.