One-to-Many Disjoint Paths in Hypercubes

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Abstract

This paper introduces a new concept called routing functions, which have a close relation to one-to-many disjoint paths in networks. By using a minimal routing function, m disjoint paths whose maximal length is minimized in the worst case can be obtained in a k-dimensional hypercube, where $m \le k$. Besides, there exist routing functions that can be used to construct m disjoint paths whose total length is minimized. The end nodes of these paths are not necessarily distinct. A minimal routing function can also be used to construct a maximal number of disjoint paths in the folded hypercube whose maximal length is minimized in the worst case.

1. Introduction

In the past decade, routing with internally node-disjoint paths (disjoint paths for short) has received much attention because disjoint paths have the advantages of efficiency and fault tolerance. There are three categories of disjoint paths, i.e., one-to-one, one-to-many, and many-to-many. Suppose that W is an interconnection network (network for short) with node connectivity k [2]. According to Menger's theorem [2], there exist k disjoint paths from one source node to another destination node in W. These disjoint paths belong to the one-to-one category. Many one-to-one disjoint paths constructed for a variety of networks can be found in the literature [3, 4, 6-10, 12, 22, 24]. There is an excellent survey of one-to-one disjoint paths in [19] where several related problems were also addressed.

According to Theorem 2.6 in [1], there exist k disjoint paths from one source node to another k distinct destination nodes in W. These disjoint paths belong to the one-to-many category. A k-dimensional hypercube (abbreviated to a k-cube) consists of 2^k nodes that are labeled with 2^k binary numbers from 0 to 2^k-1. Two nodes of a k-cube are adjacent if and only if their labels differ by exactly one bit. The node connectivity of a k-cube is k. In [23], k disjoint paths were constructed from one source node to another k destination nodes in a k-cube, where the k destination nodes were distinct. The maximal length is minimized in the worst case. In [15], m disjoint paths were constructed from one source node to another m destination nodes in a k-cube, where $m \le k$ and the m destination nodes were not necessarily distinct. The total length is minimized. One-to-many disjoint paths constructed for other networks appeared in [5, 9, 11, 20]. There were many-to-many disjoint paths constructed for the hypercube [17] and the star graph [9, 18].

In this paper, a new concept called routing functions is proposed, which is useful to derive one-to-many disjoint paths. In the next section, routing functions and their fundamental properties are introduced. In Section 3, by the aid of a minimal routing function, m disjoint paths from one source node to another m distinct destination nodes are constructed in a k-cube, where $m \le k$. It is shown in Section 4 that the maximal length of the m disjoint paths is minimized in the worst case. For any given m destination nodes, the maximal length of the resulting m disjoint paths is equal to $dis_{\max}$ or $\min\{dis_{\max}+2, k+1\}$, where $dis_{\max}$ is the maximal distance from the source node to the destination nodes. In Section 5, the situation that the m destination nodes are not necessarily distinct is discussed. In Section 6, this paper concludes with some remarks on routing functions. It is indicated that a minimal routing function can be also used to derive a maximal number of disjoint paths in the folded hypercube whose maximal length is minimized in the worst case, and there are routing functions that can be used to construct m disjoint paths in a k-cube whose total length is minimized.

2. Routing functions

Suppose that $s, d_1, d_2, \ldots, d_m$ are arbitrary $m+1$ distinct nodes of a k-cube, where $m \le k$. Since the hypercube is node symmetric, we assume $s=\overbrace{00...0}^{k}=0^k$ without loss of generality. A routing function for a k-cube is a one-to-one correspondence F from $D=\{d_1, d_2, \ldots, d_m\}$ to $I=\{n_1, n_2, \ldots, n_m\}$, where $1 \le n_j \le k$ for all $1 \le j \le m$ and $n_1, n_2, \ldots, n_m$ denote m distinct dimensions of a k-cube. Suppose $d_i=d_{i,1}d_{i,2}\ldots d_{i,k}$, and let $|d_i|$ denote the number of bits 1 contained in d_i (i.e., the distance from s to d_i), where $1 \le i \le m$. We define $V_F=(v_1, v_2, \ldots, v_k)$, where $v_j=|\{d_i \mid |d_i|=j,\ d_{i,F(d_i)}=0, \text{ and } 1 \le i \le m\}|$ for all $1 \le j \le k$. For example, if $m=k=5$, $(d_1, d_2, d_3, d_4, d_5)=$ (00011, 00101, 00111, 00110, 11100), and $(F(d_1), F(d_2), F(d_3), F(d_4), F(d_5))=(5, 2, 3, 4, 1)$, then $V_F=(0, 1, 0, 0, 0)$. We say

$(v_1, v_2, \ldots, v_k)<(v'_1, v'_2, \ldots, v'_k)$ if either $v_1<v'_1$ or $v_1=v'_1$, $v_2=v'_2, \ldots, v_{l-1}=v'_{l-1}$, and $v_l<v'_l$ for some $2\le l\le k$. F is said to be *minimal* if $V_F\le V_{F'}$ for every routing function F': $D\rightarrow I$. A minimal F can be determined in $O(k^3)$ time (see Section 6).

A minimal F is intended to derive m disjoint paths from s to $d_1, d_2, \ldots, d_m$, and there is a favorable property about a minimal F: if $d_{i_1,F(d_{i_1})}=d_{i_2,F(d_{i_2})}=\cdots=d_{i_c,F(d_{i_c})}=1$, then there exist c disjoint shortest paths from s to $d_{i_1}, d_{i_2}, \ldots, d_{i_c}$, respectively, where $\{d_{i_1}, d_{i_2}, \ldots, d_{i_c}\}\subseteq D$ (see Section 4). Let $e_b=0^{b-1}10^{k-b}$, $b=1, 2, \ldots, k$, denote the k adjacent nodes of s. An intuitive meaning of $F(d_i)=n_j$ is to assign the immediate successor of s in the path to d_i to be the node e_{n_j}.

In the rest of this section, some fundamental properties of F are introduced. Suppose that F': $D'\rightarrow I'$ and F'': $D''\rightarrow I''$ are two routing functions, where $\{D', D''\}$ is a partition of D and $\{I', I''\}$ is a partition of I. If $F'(d_i)=F(d_i)$ for all $d_i\in D'$ and $F''(d_j)=F(d_j)$ for all $d_j\in D''$, then F is said to be the *union* of F' and F'', denoted by $F=F'\cup F''$. If $F=F'\cup F''$, then $V_F=V_{F'}+V_{F''}$, i.e., $V_F=(v'_1+v''_1, v'_2+v''_2, \ldots, v'_k+v''_k)$ where $V_{F'}=(v'_1, v'_2, \ldots, v'_k)$ and $V_{F''}=(v''_1, v''_2, \ldots, v''_k)$. The following two lemmas are immediate.

Lemma 1. Suppose $F=F'\cup F''$. If $d_{i,F'(d_i)}=1$ for every $d_i\in D'$, then $V_F=V_{F''}$.

Lemma 2. Suppose that $F=F'\cup F''$ is minimal. Then F' and F'' are minimal.

Lemma 3. Suppose $d_{i,F(d_i)}=0$, $d_{j,F(d_j)}=0$, and $d_{i,F(d_j)}=1$, where $1\le i\le m$, $1\le j\le m$, and $i\ne j$. Then F is not minimal.

Proof. We define Y: $D\rightarrow I$ as follows: $Y(d_i)=F(d_j)$, $Y(d_j)=F(d_i)$, and $Y(d_r)=F(d_r)$ for all $r\in\{1, 2, \ldots, m\}-\{i, j\}$. Suppose $V_F=(v_1, v_2, \ldots, v_k)$ and $V_Y=(v'_1, v'_2, \ldots, v'_k)$. If $|d_i|\ne|d_j|$, then $v'_{|d_i|}=v_{|d_i|}-1$, $v'_{|d_j|}\le v_{|d_j|}$, and $v'_l=v_l$ for all $l\in\{1, 2, \ldots, k\}-\{|d_i|, |d_j|\}$. If $|d_i|=|d_j|$, then $v'_{|d_i|}\le v_{|d_i|}-1$ and $v'_l=v_l$ for all $1\le l\le k$ and $l\ne|d_i|$. Hence $V_Y<V_F$. □

Lemma 4. Suppose $d_{j,F(d_j)}=0$, $d_{j,F(d_i)}=1$, and $d_{i,F(d_j)}=1$, where $1\le i\le m$, $1\le j\le m$, and $i\ne j$. Then F is not minimal.

Proof. We define Y, V_Y, and V_F all the same as the above. If $d_{i,F(d_i)}=0$, then F is not minimal by Lemma 3. If $d_{i,F(d_i)}=1$, then $v'_{|d_j|}=v_{|d_j|}-1$ and $v'_l=v_l$ for all $1\le l\le k$ and $l\ne|d_j|$, which implies $V_Y<V_F$. □

Lemma 5. Suppose $d_{i,F(d_i)}=0$, $d_{i,F(d_j)}=1$, and $|d_i|<|d_j|$, where $1\le i\le m$, $1\le j\le m$, and $i\ne j$. Then F is not minimal.

Proof. We define Y, V_Y, and V_F all the same as the above. Then $v'_{|d_i|}=v_{|d_i|}-1$ and $v'_l=v_l$ for all $1\le l<|d_i|$. Hence we have $V_Y<V_F$. □

Lemma 6. If F is minimal and $d_{i,F(d_i)}=0$, then $d_{i,1}d_{i,2}\ldots d_{i,F(d_i)-1}1d_{i,F(d_i)+1}\ldots d_{i,k}\notin\{d_1, d_2, \ldots, d_m\}$ ($d_{i,1}d_{i,2}\ldots d_{i,F(d_i)-1}1d_{i,F(d_i)+1}\ldots d_{i,k}$ is the node obtained by changing the bit $d_{i,F(d_i)}$ of d_i to 1).

Proof. Suppose conversely $d_{i,1}d_{i,2}\ldots d_{i,F(d_i)-1}1d_{i,F(d_i)+1}\ldots d_{i,k}=d_r$ for some $1\le r\le m$ and $r\ne i$. That is, $d_{r,F(d_i)}=1$ and $d_{r,j}=d_{i,j}$ for all $1\le j\le k$ and $j\ne F(d_i)$. Then $F(d_r)\in\{n_1, n_2, \ldots, n_m\}$. If $d_{r,F(d_r)}=0$, then F is not minimal by Lemma 3, which is a contradiction. If $d_{r,F(d_r)}=d_{i,F(d_r)}=1$, then F is not minimal by Lemma 4, which is again a contradiction. □

3. A procedure to produce disjoint paths

In this section, a recursive procedure, named *Paths*, is proposed. With inputs minimal F, m, k, $D=\{d_1, d_2, \ldots, d_m\}$, and $I=\{n_1, n_2, \ldots, n_m\}$, the procedure can produce m disjoint shortest paths, denoted by $P_1, P_2, \ldots, P_m$, in a k-cube that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$, where P_i ($1\le i\le m$) is the path to d_i. By augmenting $P_1, P_2, \ldots, P_m$ with links (s, e_{n_1}), (s, e_{n_2}), …, (s, e_{n_m}), we have m disjoint paths from s to $d_1, d_2, \ldots, d_m$, respectively.

In the procedure, $*^{k-1}1$ and $*^{k-1}0$ represent two disjoint $(k-1)$-cubes whose nodes have rightmost bits 1 and 0, respectively, where $*\in\{0, 1\}$ and $*^{k-1}=\overbrace{**\cdots*}^{k-1}$ (usually, a k-cube is represented with $*^k$). For each node $x=x_1x_2\ldots x_k$ of a k-cube, define $x^{(k)}=x_1x_2\ldots x_{k-1}(1-x_k)$, i.e., x differs from $x^{(k)}$ only in the bit x_k. The following is a formal description of the procedure.

Procedure *Paths*(F, m, k, D, I).

Step 1. If $k=2$, then {
Construct $P_1, P_2, \ldots, P_m$ as the m disjoint paths in a 2-cube that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$.
Return $P_1, P_2, \ldots, P_m$. }

Step 2. Determine d_c so that $|d_c|\le|d_i|$ for all $1\le i\le m$, where $1\le c\le m$. If there are multiple candidates for d_c, then select arbitrary one with $d_{c,F(d_c)}=1$, or select any if they all have $d_{c,F(d_c)}=0$. Without loss of generality, we assume $c=1$, $n_c=n_1=k$, and $F(d_i)=n_i$ for all $1\le i\le m$.

Step 3. Partition D into D' and D'', where $D'=\{d_j\ |$

$d_{j,F(d_c)} = d_{j,F(d_1)} = d_{j,k} = 0$ and $1 \le j \le m\}$ and $D'' = \{d_j \mid d_{j,F(d_c)} = d_{j,F(d_1)} = d_{j,k} = 1$ and $1 \le j \le m\}$.

Step 4. Construct $P_1, P_2, \ldots, P_m$ according to the following four cases.

Case 1. $d_{1,k}=0$.
/* Without loss of generality, suppose $D'=\{d_1, d_2, \ldots, d_r\}$ and $D''=\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y': $\{d_2, d_3, \ldots, d_r\} \to \{n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$, and Y'': $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{u, n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_1^{(k)})=u$ and $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$, where $1\le u<k$ and $d_{1,u}=1$. */
Construct $P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$). /* $P_2, P_3, \ldots, P_r$ connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. */
Construct $P'_1, P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths*(Y'', $m-r+1$, $k-1$, $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{u, n_{r+1}, n_{r+2}, \ldots, n_m\}$), where P'_1 is the path to $d_1^{(k)}$. /* $P'_1, P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e_u^{(k)}, e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$. */
Construct P_1 as P'_1 augmented with the link $(d_1^{(k)}, d_1)$.
Augment $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ with links $(e_k, e_u^{(k)})$, $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$.

Case 2. $d_{1,k}=1$ and $|d_1|=1$.
/* Suppose $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y': $\{d_2, d_3, \ldots, d_r\} \to \{n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$, and Y'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */
Construct $P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$). /* $P_2, P_3, \ldots, P_r$ connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. */
Construct $P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths*(Y'', $m-r$, $k-1$, $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{n_{r+1}, n_{r+2}, \ldots, n_m\}$). /* $P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$. */
Construct P_1 as (e_k, d_1). /* P_1 has length 0. */
Augment $P_{r+1}, P_{r+2}, \ldots, P_m$ with links $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$.

Case 3. $d_{1,k}=1$, $|d_1|>1$, and $d_{1,a}=1$ for some $a \in \{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$.
/* Suppose $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y': $\{d_2, d_3, \ldots, d_r\} \to \{n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$, and Y'': $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{a, n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_1)=a$ and $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */
Construct $P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$). /* $P_2, P_3, \ldots, P_r$ connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. */
Construct $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths*(Y'', $m-r+1$, $k-1$, $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{a, n_{r+1}, n_{r+2}, \ldots, n_m\}$). /* $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e_a^{(k)}, e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$. */
Augment $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ with links $(e_k, e_a^{(k)})$, $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$.

Case 4. $d_{1,k}=1$, $|d_1|>1$, and $d_{1,a}=0$ for all $a \in \{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$.
/* Suppose $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */
Construct $P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths*(Y'', $m-r$, $k-1$, $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{n_{r+1}, n_{r+2}, \ldots, n_m\}$). /* $P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$. */
/* Define W'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $W''(d_i)=n_j$ if P_i begins at $e_{n_j}^{(k)}$ for all $r+1\le i\le m$, where $r+1\le j\le m$. */
If $d_1 \notin P_i$ for all $r+1\le i\le m$, then {

/* Define Y': $\{d_1^{(k)}, d_2, d_3, \ldots, d_r\} \to \{u, n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_1^{(k)})=u$ and $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$, where $1\le u<k$ and $d_{1,u}=1$. */
Construct $P'_1, P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths*(Y', r, $k-1$, $\{d_1^{(k)}, d_2, d_3, \ldots, d_r\}$, $\{u, n_2, n_3, \ldots, n_r\}$), where P'_1 is the path to $d_1^{(k)}$. /* $P'_1, P_2, P_3, \ldots, P_r$ connect $\{e_u, e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_1^{(k)}, d_2, d_3, \ldots, d_r\}$. */
Construct P_1 as P'_1 augmented with the link

($d_1^{(k)}$, d_1).

/* Suppose u=n_h for some r+1≤h≤m. */

Augment P_{r+1}, P_{r+2}, …, P_m with links ($e_{n_{r+1}}$, $e_{n_{r+1}}^{(k)}$), ($e_{n_{r+2}}$, $e_{n_{r+2}}^{(k)}$), …, ($e_{n_{h-1}}$, $e_{n_{h-1}}^{(k)}$), (e_k, $e_u^{(k)}$), ($e_{n_{h+1}}$, $e_{n_{h+1}}^{(k)}$), …, (e_{n_m} , $e_{n_m}^{(k)}$). }

If $d_1 \in P_l$ for some r+1≤l≤m and $d_l^{(k)} \notin$ {d_2, d_3, …, d_r}, then {

/* Define Q': { $d_l^{(k)}$, d_2, d_3, …, d_r} → {W''(d_l), n_2, n_3, …, n_r} as follows: Q'($d_l^{(k)}$)=W''(d_l) and Q'(d_i)=F(d_i)=n_i for all 2≤i≤r. */

Construct P'_l, P_2, P_3, …, P_r in $*^{k-1}0$ by executing *Paths*(Q', r, k–1, { $d_l^{(k)}$, d_2, d_3, …, d_r}, {W''(d_l), n_2, n_3, …, n_r}), where P'_l is the path to $d_l^{(k)}$. /* P'_l, P_2, P_3, …, P_r connect {$e_{W''(d_l)}$, e_{n_2} , e_{n_3} , …, e_{n_r} } and { $d_l^{(k)}$, d_2, d_3, …, d_r}. */

Construct P_1 as the subpath of P_l from $e_{W''(d_l)}^{(k)}$ to d_1.

Reconstruct P_l as P'_l augmented with the link ($d_l^{(k)}$, d_l).

Augment P_1, P_{r+1}, P_{r+2}, …, P_{l-1}, P_{l+1}, …, P_m with links ($e_{n_{r+1}}$, $e_{n_{r+1}}^{(k)}$), ($e_{n_{r+2}}$, $e_{n_{r+2}}^{(k)}$), …, ($e_{W''(d_l)-1}$, $e_{W''(d_l)-1}^{(k)}$), (e_k, $e_{W''(d_l)}^{(k)}$), ($e_{W''(d_l)+1}$, $e_{W''(d_l)+1}^{(k)}$), …, (e_{n_m} , $e_{n_m}^{(k)}$). }

If $d_1 \in P_l$ for some r+1≤l≤m and $d_l^{(k)} \in$ {d_2, d_3, …, d_r}, then

/* Suppose $d_l^{(k)}$ =d_h for some 2≤h≤r. Define Y: {d_1, d_2, …, d_m} → {n_1, n_2, …, n_m} as follows: Y(d_l)=F(d_h), Y(d_h)=W''(d_l), Y(d_1)=F(d_1)=k, Y(d_i)=F(d_i) for all 2≤i≤r and i≠h, and Y(d_j)=W''(d_j) for all r+1≤j≤m and j≠l. */

Construct P_1, P_2, …, P_m all the same as Case 3. /* Substitute Y for the F in Case 3. */

Step 5. Return P_1, P_2, …, P_m.

Intuitively, the m disjoint paths from s to d_1, d_2, …, d_m can be obtained by first including m links (s, $e_{F(d_1)}$), (s, $e_{F(d_2)}$), …, (s, $e_{F(d_m)}$) and then constructing m disjoint paths, i.e., P_1, P_2, …, P_m, that connect {$e_{F(d_1)}$, $e_{F(d_2)}$, …, $e_{F(d_m)}$ } and {d_1, d_2, …, d_m}. The latter can be done by executing *Paths*(F, m, k, D, I). In Step 2, the k-cube was divided into two (k–1)-cubes $*^{k-1}0$ and $*^{k-1}1$, according to the dimension F(d_c)=k. In Step 3, {d_1, d_2, …, d_m} was partitioned into D' and D'' which contain nodes belonging to $*^{k-1}0$ and $*^{k-1}1$, respectively. In Step 4, P_1, P_2, …, P_m were constructed according to four cases in which paths in $*^{k-1}0$ and $*^{k-1}1$ need to be constructed in a recursive manner. In Case 1, |D'|–1 disjoint paths in $*^{k-1}0$ that connect { $e_{F(d_2)}$, $e_{F(d_3)}$, …, $e_{F(d_r)}$ } and D'–{d_1} and |D''|+1 disjoint paths in $*^{k-1}1$ that connect { $e_u^{(k)}$, $e_{F(d_{r+1})}^{(k)}$, $e_{F(d_{r+2})}^{(k)}$, …, $e_{F(d_m)}^{(k)}$ } and D''∪{ $d_1^{(k)}$ } were constructed. It should be noted that e_k in $*^{k-1}1$ corresponds to s=0^k in $*^{k-1}0$ (refer to Figure 1). In Case 2, |D'| disjoint paths in $*^{k-1}0$ that connect { $e_{F(d_2)}$, $e_{F(d_3)}$, …, $e_{F(d_r)}$ } and D' and |D''|–1 disjoint paths in $*^{k-1}1$ that connect { $e_{F(d_{r+1})}^{(k)}$, $e_{F(d_{r+2})}^{(k)}$, …, $e_{F(d_m)}^{(k)}$ } and D''–{d_1} were constructed. Since d_1=e_k, P_1 has length 0. In Case 3, |D'| disjoint paths in $*^{k-1}0$ that connect { $e_{F(d_2)}$, $e_{F(d_3)}$, …, $e_{F(d_r)}$ } and D' and |D''| disjoint paths in $*^{k-1}1$ that connect { $e_a^{(k)}$, $e_{F(d_{r+1})}^{(k)}$, $e_{F(d_{r+2})}^{(k)}$, …, $e_{F(d_m)}^{(k)}$ } and D'' were constructed.

In Case 4, |D''|–1 disjoint paths in $*^{k-1}1$ that connect { $e_{F(d_{r+1})}^{(k)}$, $e_{F(d_{r+2})}^{(k)}$, …, $e_{F(d_m)}^{(k)}$ } and D''–{d_1} were first constructed. The subsequent construction depends on whether d_1 is contained in these |D''|–1 paths or not. If d_1 is not contained in these |D''|–1 paths, then |D'|+1 disjoint paths in $*^{k-1}0$ that connect {e_u, $e_{F(d_2)}$, $e_{F(d_3)}$, …, $e_{F(d_r)}$ } and D'∪{ $d_1^{(k)}$ } were constructed. If d_1 is contained in one of these |D''|–1 paths which connects $e_{W''(d_l)}^{(k)}$ and d_l and $d_l^{(k)} \notin D'$, then |D'|+1 disjoint paths in $*^{k-1}0$ that connect { $e_{W''(d_l)}$, $e_{F(d_2)}$, $e_{F(d_3)}$, …, $e_{F(d_r)}$ } and D'∪{ $d_l^{(k)}$ } were constructed. If d_1 is contained in one of these |D''|–1 paths that connects $e_{W''(d_l)}^{(k)}$ and d_l and $d_l^{(k)} \in D'$, then after substituting Y for F, the situation is the same as Case 3 and P_1, P_2, …, P_m can be obtained likewise.

The maximal length of the m disjoint paths from s to d_1, d_2, …, d_m is computed as follows.

Theorem 1. Suppose that s, d_1, d_2, …, d_m are arbitrary m+1 distinct nodes of a k-dimensional hypercube, where m≤k and k≥2. There are m disjoint paths from s to d_1, d_2, …, d_m, respectively, whose maximal length is not greater than k+1 if m=k, and not greater than k if m<k. The maximal length is minimized in the worst case.

The proof of Theorem 1 is presented in the next section.

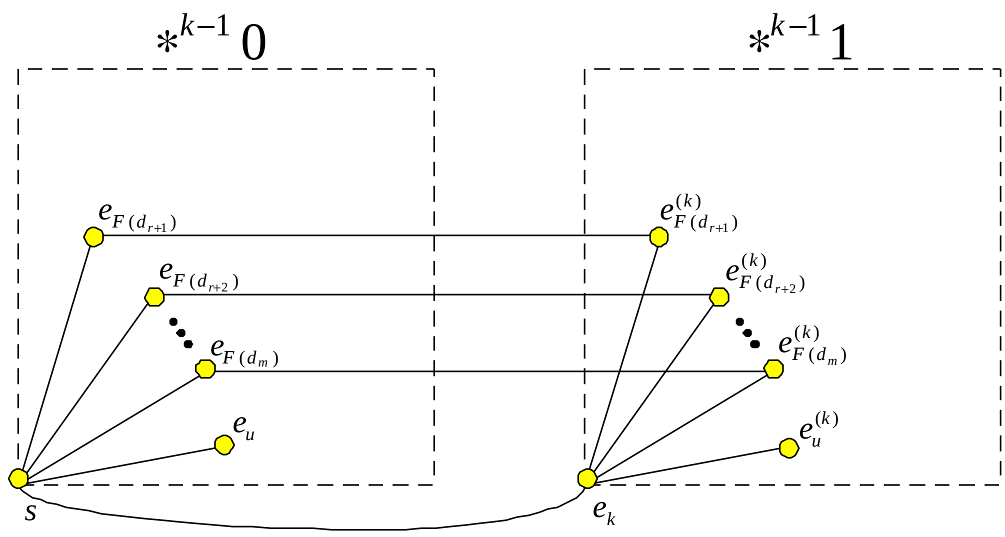

Figure 1. e_k in $*^{k-1}1$ corresponds to $s=0^k$ in $*^{k-1}0$.

4. Proof of Theorem 1

We need to show two properties of $P_1, P_2, \ldots, P_m$: (P1) $P_1, P_2, \ldots, P_m$ are disjoint, and (P2) for all $1 \le i \le m$, if $d_{i,F(d_i)}=1$, then P_i has length $|d_i|-1$, and if $d_{i,F(d_i)}=0$, then P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$, where t_i is the immediate predecessor of d_i in P_i. According to (P2), P_i has length $k-1$ if $|d_i|=k$, and at most k if $|d_i|<k$. Hence, P_i has length at most k. However, when $m<k$, P_i has length at most $k-1$, as explained below. If $1^k \in \{d_1, d_2, \ldots, d_m\}$, then P_i has length at most $k-1$, for otherwise (P_i has length k) we have $|t_i|=|d_i|+1=(k-1)+1=k$ (i.e., $t_i=1^k$) according to (P2). This contradicts to (P1). On the other hand, if $1^k \notin \{d_1, d_2, \ldots, d_m\}$, then after adding a new node $d_{m+1}=1^k$ to D and a new dimension $u \in \{1, 2, \ldots, k\}-I$ to I, *Paths*$(F, m+1, k, D, I)$ can produce $P_1, P_2, \ldots, P_{m+1}$. Since $1^k \in \{d_1, d_2, \ldots, d_{m+1}\}$, $P_1, P_2, \ldots, P_{m+1}$ have lengths at most $k-1$.

Consequently, the m disjoint paths from s to $d_1, d_2, \ldots, d_m$ have maximal length $k+1$ if $m=k$, and k if $m<k$. Since the diameter of a k-cube is k, the maximal length of the m disjoint paths is at least k in the worst case. It was shown in [11] that the maximal length is at least $k+1$ in the worst case when $m=k$. Hence the maximal length in Theorem 1 is minimized in the worst case.

(P1) and (P2) can be verified by induction on k. The detailed proof my be found in [21].

5. When $\{d_1, d_2, \ldots, d_m\}$ is a multiset

A *multiset* is a collection of elements in which multiple occurrences of the same element are allowed [16]. *Paths*(F, m, k, D, I) can deal with a multiset D, if Step 4 is modified as follows. Let $T=\{d_j \mid d_j=e_k \text{ and } r+1 \le j \le m\}$. In Case 2, Y'' is changed to Y'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\}-T \rightarrow \{n_{r+1}, n_{r+2}, \ldots, n_m\}-\{F(d_j) \mid d_j \in T \text{ and } r+1 \le j \le m\}$ so that $Y''(d_i)=F(d_i)=n_i$ for all $r+1 \le i \le m$ and $d_i \ne e_k$. Instead of executing *Paths*$(Y'', m-r, k-1, \{d_{r+1}, d_{r+2}, \ldots, d_m\}, \{n_{r+1}, n_{r+2}, \ldots, n_m\})$, we execute *Paths*$(Y'', m-r-|T|, k-1, \{d_{r+1}, d_{r+2}, \ldots, d_m\}-T, \{n_{r+1}, n_{r+2}, \ldots, n_m\}-\{F(d_j) \mid d_j \in T \text{ and } r+1 \le j \le m\})$, in order to construct $m-r-|T|$ disjoint paths in $*^{k-1}1$ that connect $\{e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\}-\{e^{(k)}_{F(d_j)} \mid d_j \in T \text{ and } r+1 \le j \le m\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}-T$. Additionally, we construct another $|T|$ disjoint paths of length one that connect $e^{(k)}_{F(d_j)}$ and d_j for all $d_j \in T$.

In Case 4, the conditions for the second and third situations are changed to "$d_1 \in P_l$ for some $r+1 \le l \le m$, $d_1 \notin \{d_{r+1}, d_{r+2}, \ldots, d_m\}$, and $d^{(k)}_l \notin \{d_2, d_3, \ldots, d_r\}$" and "$d_1 \in P_l$ for some $r+1 \le l \le m$, $d_1 \notin \{d_{r+1}, d_{r+2}, \ldots, d_m\}$, and $d^{(k)}_l \in \{d_2, d_3, \ldots, d_r\}$", respectively. One more situation whose condition is "$d_1 \in P_l$ for some $r+1 \le l \le m$ and $d_1 \in \{d_{r+1}, d_{r+2}, \ldots, d_m\}$" needs to be added. The new situation constructs $P_1, P_2, \ldots, P_r$ in $*^{k-1}0$ all the same as the first situation (i.e., $d_1 \notin P_i$ for all $r+1 \le i \le m$).

Both (P1) and (P2) remains correct after the modifications above, as shown in [21]. When $\{d_1, d_2, \ldots, d_m\}$ is a multiset, Theorem 1 can be rewritten as follows.

Theorem 2. Suppose that $s, d_1, d_2, \ldots, d_m$ are arbitrary $m+1$ nodes of a k-dimensional hypercube so that $s \notin \{d_1, d_2, \ldots, d_m\}$ and $\{d_1, d_2, \ldots, d_m\}$ is a multiset, where $m \le k$ and $k \ge 2$. There are m disjoint paths from s to $d_1, d_2, \ldots, d_m$, respectively, whose maximal length is not greater than $k+1$ if $m=k$, and not greater than k if $m<k$. The maximal length is minimized in the worst case.

6. Discussion and conclusion

The main contribution of this paper is to show the effectiveness of routing functions in deriving one-to-many disjoint paths in networks. By using a minimal routing function, m disjoint paths whose maximal length is minimized in the worst case can be obtained in a k-cube, where $m \le k$. The problem of finding a minimal routing function can be reduced to the problem of finding a maximum matching in a weighted bipartite graph. Suppose $G=(U, V, E)$ is a weighted bipartite graph, where U and V are two partite sets of nodes and E is the set of weighted links (each link of G is assigned a weight). A subset of E forms a *matching* in G if no two of them share a common node. A matching in G is *maximum* if its total weight is maximum. A maximum matching in G can be found in $O(|U \cup V|(|E|+|U \cup V|\log|U \cup V|))$ time (see [14]).

Suppose F: $\{d_1, d_2, \ldots, d_m\} \rightarrow \{n_1, n_2, \ldots, n_m\}$ is a routing function, where $\{d_1, d_2, \ldots, d_m\}$ is a multiset. Let $U=\{d_1, d_2, \ldots, d_m\}$, $V=\{n_1, n_2, \ldots, n_m\}$, and $E=\{(d_i, n_j) \mid 1 \le i \le m$ and $1 \le j \le m\}$. We assign each link (d_i, n_j) a weight $(k+1)^{k-|d_i|+1}$ if $d_{i,n_j}=1$, and 1 if $d_{i,n_j}=0$. A maximum matching M in G contains m links and its total weight can be expressed as $a_1*(k+1)^k+a_2*(k+1)^{k-1}+ \ldots +a_k*(k+1)+a_{k+1}$, where $0 \le a_r \le m$ for all $1 \le r \le k+1$. That is, there are a_r links in M whose weights are $(k+1)^{k-r+1}$ (or equivalently, M contains a_v links (d_i, n_j) of weight $(k+1)^{k-v+1}$ with $|d_i|=v$ and $d_{i,n_j}=1$ for all $1 \le v \le k$, and a_{k+1} links (d_i, n_j) of weight 1 with $d_{i,n_j}=0$). For all $1 \le v \le k$, let r_v denote the number of d_i's with $|d_i|=v$ and $d_i \in U$. A minimal F with $V_F=(r_1-a_1, r_2-a_2, \ldots, r_k-a_k)$ can be obtained from M as follows: $F(d_i)=n_j$ if $(d_i, n_j) \in M$. If F is not minimal, then there exists $V_{F'}<V_F$ for some routing function F': $\{d_1, d_2, \ldots, d_m\} \rightarrow \{n_1, n_2, \ldots, n_m\}$, which implies another matching in G whose total weight is greater than the total weight of M. This is a contradiction.

Besides minimal routing functions, there are some other routing functions that can be used to produce disjoint paths with different properties. For example, m disjoint paths whose total length is minimized can be also produced in a k-cube, if an implicit routing function in [15] is used. In [15], m non-empty subsets $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, k\}$ were used to represent m nodes $d_1, d_2, \ldots, d_m$, respectively, so that for all $1 \le u \le m$ and $1 \le w \le k$, $w \in X_u$ if and only if $d_{u,w}=1$. A set of c distinct integers $t_1 \in X_{h_1}$, $t_2 \in X_{h_2}$, $\ldots$, $t_c \in X_{h_c}$ is called a *partial System of Distinct Representatives* (*SDR* for short) for $\{X_1, X_2, \ldots, X_m\}$ if $h_1, h_2, \ldots, h_c$ are all distinct, where $c \le m$ and $1 \le h_i \le m$ for all $1 \le i \le c$. Further, the partial SDR $\{t_1, t_2, \ldots, t_c\}$ is *maximum* if c is maximized. When $c=m$, $\{t_1, t_2, \ldots, t_c\}$ is called an *SDR*. A maximum partial SDR $\{t_1, t_2, \ldots, t_c\}$ can be used to construct m disjoint paths in a k-cube whose total length is minimized, if there is no $j \in \{1, 2, \ldots, m\}-\{h_1, h_2, \ldots, h_c\}$ satisfying the following two conditions: (C1) $X_j \subset X_{h_i}$ for some $1 \le i \le c$, and (C2) there exists an SDR for $\{X_{h_1}, X_{h_2}, \ldots, X_{h_{i-1}}, X_j, X_{h_{i+1}}, \ldots, X_{h_c}\}$. Such a maximum partial SDR can be determined in $O(k^{2.5})$ time (see [15]).

Actually, a maximum partial SDR $\{t_1, t_2, \ldots, t_c\}$ can be regarded as a routing function F: $D \rightarrow I$ so that $\{t_1, t_2, \ldots, t_c\} \subseteq I$ and $F(d_{h_i})=t_i$ for all $1 \le i \le c$. Suppose $L_F=\{u \mid d_{u,F(d_u)}=1$ and $1 \le u \le m\}$. It follows that $Paths(F, m, k, D, I)$ can result in m disjoint paths from s to $d_1, d_2, \ldots, d_m$ whose total length is minimized, provided $|L_F|$ is maximized and there is no $j \in \{1, 2, \ldots, m\}-L_F$ satisfying the following two conditions: (C1′) $d_j \ne d_l$ and $d_{j,w} \le d_{l,w}$ for some $l \in L_F$ and all $1 \le w \le k$, and (C2′) there exists a routing function Y with $d_{j,Y(d_j)}=d_{v,Y(d_v)}=1$ for all $v \in L_F-\{l\}$. Here, a routing function F: $D \rightarrow I$ with maximum $|L_F|$ corresponds to a maximum partial SDR (L_F corresponds to $\{h_1, h_2, \ldots, h_c\}$ and $\{F(d_u) \mid u \in L_F\}$ corresponds to $\{t_1, t_2, \ldots, t_c\}$). Besides, (C1′) and (C2′) correspond to (C1) and (C2), respectively. A routing function F with maximum $|L_F|$ so that there is no $j \in \{1, 2, \ldots, m\}-L_F$ satisfying (C1′) and (C2′) can be determined with the same time complexity as a maximum partial SDR so that there is no $j \in \{1, 2, \ldots, m\}-\{h_1, h_2, \ldots, h_c\}$ satisfying (C1) and (C2). There exist other routing functions that can result in m disjoint paths from s to $d_1, d_2, \ldots, d_m$ whose total length is minimized. For example, if (C1′) is changed to "$|d_j|<|d_l|$ for some $l \in L_F$", then the resulting F also serves the purpose.

We have shown in Theorem 2 that the maximal length of the m disjoint paths from s to $d_1, d_2, \ldots, d_m$ is minimum for the worst-case scenario. According to (P2), each path from s to d_i has length $|d_i|$ (i.e. shortest) if $d_{i,F(d_i)}=1$, and $|d_i|+2$ (i.e., second shortest) if $d_{i,F(d_i)}=0$, where $1 \le i \le m$. It follows that for any given $d_1, d_2, \ldots, d_m$, the maximal length of the m disjoint paths from s to $d_1, d_2, \ldots, d_m$ is equal to $\max\{|d_i| \mid 1 \le i \le m\}$ or $\min\{\max\{|d_i| \mid 1 \le i \le m\}+2, k+1\}$. It should be noted that $P_1, P_2, \ldots, P_m$ are all shortest. The node $e_{F(d_i)}$ (or $e_{Y(d_i)}$ for the situation of $d_1 \in P_l$ for some $r+1 \le l \le m$ and $d_l^{(k)} \in \{d_2, d_3, \ldots, d_r\}$ in Case 4 of Step 4) is the immediate successor of s in the path to d_i. When $d_{i,F(d_i)}=1$, $e_{F(d_i)}$ (or $e_{Y(d_i)}$) is contained in a shortest path from s to d_i. When $d_{i,F(d_i)}=0$, $e_{F(d_i)}$ is not contained in any shortest path from s to d_i.

A k-dimensional folded hypercube [13] is basically a k-cube augmented with 2^{k-1} complement links. It was shown in [21] that using a minimal routing function, $k+1$ disjoint paths whose maximal length is minimized in the worst case can be constructed in a k-dimensional folded hypercube, where $k+1$ is the node connectivity. It is worth while exploring more relations between the characteristics of routing functions and the properties of disjoint paths.

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