Trimming Curve Approximation for Trimmed Surfaces

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Abstract

A trimmed surface is defined to be a parametric surface together with trimming curves lying in the parametric space **D** of the surface. This paper investigates the interrelation between surface tessellation and trimming curve approximation, and shows that existing trimmed surface tessellation algorithms have some problems on trimming curve approximation. Several examples are proposed to show that a valid approximation of trimming curves in **D** together with the refinement imposed by surface tessellation does not necessarily generate a valid linear approximation in 3D space. Then we propose a novel step-length estimation method such that a piecewise linear interpolant of the trimming curve based on proposed step length will assure the 3D derivation tolerance. In this method, we exploit the triangle inequality and take the derivation tolerance in 3D space into account to compute the effective step length. Moreover, several empirical tests are given to demonstrate the correctness of our step length estimation.

Keywords: Trimmed Surface, Tessellation

1 Introduction

Parametric surfaces, such as non-uniform rational B-spline surfaces (NURBs), have been widely used in geometric modeling and also been proposed as one of standard primitives for computer aided design(CAD) systems. Moreover, as algorithms in computer aided geometric design are incorporated in various design systems, the trimmed parametric surfaces have become a fundamental building block for surface modeling [3, 4].

A trimmed surface is composed of two major components, namely, a tensor product parametric surface and a set of properly oriented trimming curves. In order to accommodate various CAD processes such as model visualization, cutter path generation and area computations, the trimmed surface is commonly discretized and approximated by triangular facets through tessellation algorithms. These tessellation algorithms involve choosing step lengths along each parameter to control the closeness between the resultant tessellants and the surface. In general, the step lengths are derived from approximation criteria such as chordal length [1, 7, 8, 10, 11] and chordal derivation [1, 8, 9, 12] .

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The chordal length criterion requires that the length of each edge of the tessellant is less than the given tolerance. Usually it is used for the purpose of real time rendering. The chordal derivation criterion is to ensure the distance between the tessellants (traingles or other polygons) and the surface is less than the given tolerance. Besides, it requires a tessellation that can produce a minimum number of approximation polygons. As we are interested mainly in the approximation of trimming surface, we confine ourselves to the step length determination problems subjected to chordal derivation.

The remainder of this paper is organized as follows. Section 2 addresses the problems of approximating the trimming curve in existing trimmed surface tessellation algorithms. Then Section 3 proposes a step length estimation method respectively such that the trimming curve approximation together with the surface tessellation in **D** will result in valid linear approximation in 3D space. Moreover, in the Section 4, empirical results are given to demonstrate the feasibility and the correctness of our estimation. A final conclusion will be given in Section 5.

2 The Problems

Since the tessellation of 3D models is of great interest for many applications such as rendering, machining and finite element computations, several tessellation algorithms for trimmed surfaces have been proposed [2, 6, 7, 8, 9, 11, 12]. However, most existing tessellation algorithms concentrate on controlling the chordal derivation between the surface and the approximated triangles. In this study, we will point out that the criterion based on controlling 2D derivation error of trimming curves and 3D derivation error of surface approximation does not imply the error of trimming curve approximation is under control in 3D space. Besides, we propose a novel step length estimation for trimming curve tessellation to remedy this flaw.

2.1 Step Length for Surface Tessellation

Suppose $S(u,v)$ is an order $m*n$ degree tensor product rational parametric surface, defined in the domain **D**=

$[0,1]\times[0,1]$, such that

$$\begin{aligned} S(u,v) &= (x(u,v),y(u,v),z(u,v)) \\ &= \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j} r_{i,j} \phi_j(u)\varphi_i(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}\phi_j(u)\varphi_i(v)} \end{aligned} \quad (1)$$

where $r_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})$ are the control points, $w_{i,j}$ are associated weights, and $\phi_j(u)$ and $\varphi_i(v)$ can be Bernstein-Bézier polynomials or B-spline basis functions.

If we triangulate the domain **D** with triangles, then the set of all 3D triangles obtained by mapping the 2D ones onto the surface forms a linear interpolant of $S(u,v)$. The step length problem is to seek a threshold Δ such that if the edge length of any 2D triangle is less than Δ, then the corresponding 3D triangle deviates from the surface is less than given tolerance.

There has been several different solutions to the step length problem [1, 8, 9, 12]. For exposition purpose, we choice the one proposed in [1, 5]. According to [5], if **D** is uniformly tessellated with step lengths Δ_u and Δ_v along parameters u and v respectively, then the derivation between the surface and the tessellant is bounded by the following estimation,

$$E(\Delta_u,\Delta_v) = \frac{1}{8}(\Delta_u^2 M^{uu} + 2\Delta_u\Delta_v M^{uv} + \Delta_v^2 M^{vv}). \quad (2)$$

where M^{uu}, M^{uv} and M^{vv} are the sup-norms for the second order partial derivative of surface. We can set $E(\Delta_u,\Delta_v) = \epsilon$ and try to solve the step lengths Δ_u and Δ_v in equation (2). However, there is only one equation for two unknowns. To cope with this problem, we follow the suggestion in [1] by requiring the function $2\frac{1}{\Delta_u}\frac{1}{\Delta_v}$ (the number of triangles) to be minimized. Then, by introducing the Lagrange's multiplier λ, we can define $m(\Delta_u,\Delta_v)$ to be

$$(\frac{2}{\Delta_u\Delta_v}) + \lambda(\frac{1}{8}(\Delta_u^2 M^{uu} + 2\Delta_u\Delta_v M^{uv} + \Delta_v^2 M^{vv}) - \epsilon).$$

It is known that the extreme value occurs when

$$\begin{aligned} \frac{\partial m}{\partial \Delta_u} &= \frac{-2}{\Delta_v\Delta_u^2} + \frac{\lambda}{4}(\Delta_u M^{uu} + \Delta_v M^{uv}) = 0, \;\; and \\ \frac{\partial m}{\partial \Delta_v} &= \frac{-2}{\Delta_u\Delta_v^2} + \frac{\lambda}{4}(\Delta_u M^{uv} + \Delta_v M^{vv}) = 0. \end{aligned} \quad (3)$$

From (2) and (3), we have three unknowns and three equations. By eliminating λ from (3), we have $\Delta_v = \sqrt{\frac{M^{uu}}{M^{vv}}}\Delta_u$, which is then substituted into (2) to obtain,

$$\begin{aligned} \Delta_u &= \frac{2\sqrt{\epsilon\ M^{vv}}}{\sqrt{M^{uu}M^{vv} + M^{uv}\sqrt{M^{uu}M^{vv}}}}, \;\; and \\ \Delta_v &= \frac{2\sqrt{\epsilon\ M^{uu}}}{\sqrt{M^{uu}M^{vv} + M^{uv}\sqrt{M^{uu}M^{vv}}}}. \end{aligned} \quad (4)$$

2.2 Step Length for Trimming Curve Approximation

In general,a trimming curve is obtained either by solving surface/surface intersection problem or being specified directly as a piecewise parametric curve in **D**. In this study, a trimming curve segment is described by its parametric form as follows,

$$\begin{aligned} \mathsf{c}(t) &= (u(t), v(t)) \\ &= \frac{\sum_{i=0}^{n} w_i p_i \phi_i(t)}{\sum_{i=0}^{n} w_i \phi_i(t)}, \quad t \in [0,1] \end{aligned} \quad (5)$$

where $p_i = (u_i, v_i)$ are the control points and n is the degree of curve. Note that, the weight and basis functions of trimming curve are of the same type as that of the surface. That is, if the surface is a Bézier or B-spline surface, then the trimming curve is also of Bézier or B-spline type.

In most existing tessellation algorithms [1, 9, 12], the trimming curve $c(t)$ is usually further approximated by a piecewise linear function. A typical approach to tessellate $c(t)$ into piecewise linear functions is briefly described in the following paragraph. Then we show by counter examples that this tessellation does not assure the chordal derivation error in 3D modeling space.

Without loss of generality, we assume the trimming curve is a single curve segment $c(t)$ hereafter. This is because each curve segment can be approximated separately. Let $\epsilon > 0$ be the given tolerance. According to the estimation of chordal derivation given in [5], we uniformly partition $I_0 = [0,1]$ into n subintervals by tessellation points $\{t_i = \frac{i}{n} \mid i = 0,...,n\}$ where $n = \lceil\sqrt{\sup_{t\in I_0}\frac{\|c"(t)\|}{8\epsilon}}\rceil$. Then, the piecewise linear function $l(t) = \{\overline{P_{i-1}P_i} \mid P_i = c(t_i), i = 1,...,n\}$ satisfies $\|c(t) - l(t)\|_\infty < \epsilon$ [1, 5, 12]. In other words, $l(t)$ is a valid approximation to $c(t)$ in **D** within ϵ. However, if we lift each vertex P_i in **D** to $P_i^* = S(P_i)$ in $\mathbf{R^3}$ and form the polygon $L(t) = \{\overline{P_{i-1}^*P_i^*}\}$ called lifted polygon for brevity, then $L(t)$ may not be a valid linear approximation of the 3D trimming curve $\Gamma(t) = S(c(t))$ within ϵ. Example in Table 1 shows that $L(t)$ fails to approximate $\Gamma(t)$ within given tolerance. In this example, $\|l(t) - c(t)\|_\infty = .008295$, which is less than the tolerance $\epsilon = .01$, but $\|L(t) - \Gamma(t)\|_\infty = .012213$, which exceeds the tolerance ϵ. Therefore, the tessellation based on $c(t)$ only is not sufficient to yield a valid tessellation for 3D trimming curve $\Gamma(t)$.

There is another problem which is overlooked by existing algorithms in computing the tessellation of trimming curves. Suppose the trimming curve $c(t)$ is approximated by a linear interpolant $l(t) = \{\overline{P_{i-1}P_i}\}$, where $P_i = c(t_i)$, such that the lifted polygon $L(t)$ do approximate $\Gamma(t)$ correctly within given tolerance. It is known that the tessellation of a trimmed parametric surface is accomplished by triangulating the corresponding region in **D** [1, 7, 9, 12]. In order to triangulate the desired region along the curve $c(t)$ properly, the surface tessellation algorithm has to find intersection points of surface tessellation lines and $l(t)$ to serve as vertices of triangles mesh (see Figure 4). Intuitively, $l(t)$ is further tessellated by superimposing the surface tessellation lines on $l(t)$. Let $\{Q_j\}$ be the set of intersection

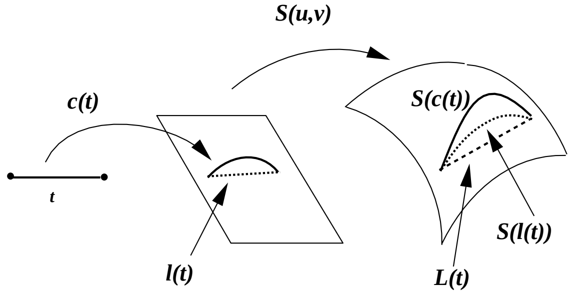

Fig. 1: Chordal derivations of the trimming curve in **D** domain and 3D space

points of $l(t)$ and surface tessellation lines. Then the super set $\{R_k\} = \{P_i\} \bigcup \{Q_j\}$ induces a refined polygon $l_{\oplus}(t) = \{\overline{R_{j-1}R_j}\}$ of $l(t)$, and the corresponding lifted polygon $L_{\oplus}(t) = \{\overline{R^*_{j-1}R^*_j} \mid R^*_j = S(R_j) \in \mathbf{R^3}\}$ becomes the final linear approximation to $\Gamma(t)$. However, example in of Table 2 shows that $L_{\oplus}(t)$ does not yield a valid approximation to $\Gamma(t)$. In this example, the tolerance ϵ is .087 and $\|l(t) - c(t)\|_\infty = .0661$, but $\|L_{\oplus}(t) - \Gamma(t)\|_\infty = .091997$ which exceeds the tolerance ϵ. Therefore, we also need a linear approximation of the trimming curve, which is robust against triangulation.

3 The Approach Based on Triangle Inequality

This section looks for proper step length Δ_t which tessellates $I_0 = [0,1]$ into n subintervals $I_i = [t_{i-1}, t_i]$ (where $t_0 = 0, t_i = t_{i-1} + \Delta_t, i = 1, ..., n$) such that the piecewise linear function $L(t) = \{\overline{P^*_{i-1}P^*_i} | P^*_i = \Gamma(t_i), i = 0, ..., n\}$ satisfies $|\Gamma(t) - L(t)| < \epsilon$. Moreover, the linear interpolant $l(t)$ of $c(t)$ is robust against triangulation. In other words, if $l(t)$ is refined by the surface tessellation into $l_{\oplus}(t)$ described in the last section, then $|\Gamma(t) - L_{\oplus}(t)| < \epsilon$, where $L_{\oplus}(t)$ is the lifted polygon of $l_{\oplus}(t)$.

First let us consider the following triangle inequality for $t \in [t_{i-1}, t_i]$, (see Figure 1)

$$\begin{aligned} &|\Gamma(t) - L(t)| = |S(c(t)) - L(t)| \\ &\le |S(c(t)) - S(l(t))| + |S(l(t)) - L(t)| \end{aligned} \tag{6}$$

According to Mean Value Theorem, the first term on the right hand side of (6) is bounded by

$$|S(c(t)) - S(l(t))| \le |D_{c(t)-l(t)}(\mathsf{S}(a,b))| \cdot |c(t) - l(t)|$$

for some point (a, b) in the line segment connecting $c(t)$ and $l(t)$, and $D_{c(t)-l(t)}(S(a,b))$ is the directional derivative of S along the vector $c(t) - l(t)$ at (a, b). Since

$$|D_{c(t)-l(t)}(S(a,b))| \le |\frac{\partial S(a,b)}{\partial u}| + |\frac{\partial S(a,b)}{\partial v}|$$

it follows that

$$|S(c(t)) - S(l(t))| \le (M^u + M^v)|c(t) - l(t)|$$

where M^u and M^v are sup-norms of the first order partial derivatives of $S(u, v)$. Therefore, by applying the Filip's estimation [5] to $|c(t) - l(t)|$ with step length δ_1, we have

$$|S(c(t)) - S(l(t))| \le \frac{1}{8}\delta_1^2(M^u + M^v)M^{tt} \tag{7}$$

where $M^{tt} = \|c^{"}(t)\|_\infty$.

Moreover, the second term on the right hand side of (6) can also be bounded by

$$\begin{aligned} &|S(l(t)) - L(t)| \\ &\le \frac{1}{8}\delta_2^2 \sup |\frac{d^2}{dt^2}S(l(t))| \\ &= \frac{1}{8}\delta_2^2 \sup |l^{'}(t)H_S l^{'}(t)^T + D(S(l(t)))l^{"}(t)|, \end{aligned}$$

where $H_s = \begin{bmatrix} \frac{\partial^2 S(u,v)}{\partial u^2} & \frac{\partial^2 S(u,v)}{\partial u \partial v} \\ \frac{\partial^2 S(u,v)}{\partial u \partial v} & \frac{\partial^2 S(u,v)}{\partial v^2} \end{bmatrix}$ is the Hessian matrix of S, and D is the derivative of S. Since $l(t)$ is a linear interpolant of $c(t) = (u(t), v(t))$, $l^{'}(t) = (p, q)$ is a constant vector and $l^{"}(t) = (0,0)$ for $t \in [t_{i-1}, t_i]$. Besides, $|p| \le \max_{t\in[t_{i-1},t_i]} |u^{'}(t)| \le \max_{t\in[0,1]} |u^{'}(t)|$ and $|q| \le \max_{t\in[t_{i-1},t_i]} |v^{'}(t)| \le \max_{t\in[0,1]} |v^{'}(t)|$, which imply

$$\begin{aligned} &|S(l(t)) - L(t)| \le \frac{1}{8}\delta_2^2 \sup |l^{'}(t)H_S l^{'}(t)^T| \\ &\le \frac{1}{8}\delta_2^2(M^{uu}(M_u^t)^2 + 2M^{uv}M_u^tM_v^t + M^{vv}(M_v^t)^2), \end{aligned} \tag{8}$$

where $M^{uu} = \|\frac{\partial^2 S(u,v)}{\partial u^2}\|_\infty$, $M^{uv} = \|\frac{\partial^2 S(u,v)}{\partial u \partial v}\|_\infty$, $M^{vv} = \|\frac{\partial^2 S(u,v)}{\partial v^2}\|_\infty$, $M_u^t = \max_{t\in[0,1]} |u'(t)|$ and $M_v^t = \max_{t\in[0,1]} |v'(t)|$.

From (7) and (8), we have

$$|S(c(t)) - L(t)| \le (M_1\delta_1^2 + M_2\delta_2^2) \tag{9}$$

where $M_1 = \frac{1}{8}(M^u + M^v)M^{tt}$ and $M_2 = \frac{1}{8}(M^{uu}(M_u^t)^2 + 2M^{uv}M_u^tM_v^t + M^{vv}(M_v^t)^2)$.

If we set

$$M_1\delta_1^2 = \lambda\epsilon \ \ and \ \ M_2\delta_2^2 = (1-\lambda)\epsilon \quad (0 < \lambda < 1), \tag{10}$$

then from (9) the chordal derivation $|S(c(t)) - L(t)|$ is bounded by given ϵ.

Now it remains to find an optimal λ so that we have the longest step length. By simply observing the graph of two parabolas in (10), we know that the optimal tessellation, that is $\Delta_t = \max_{\lambda\in(0,1)}(\min(\delta_1, \delta_2))$, happens when two parabolas intersect. That is $\frac{\lambda\epsilon}{M_1} = \frac{(1-\lambda)\epsilon}{M_2}$, which gives $\lambda = \frac{M_1}{M_1+M_2}$. Substituting the value of λ into (10), we have the desired step length

$$\Delta_t = \sqrt{\frac{\epsilon}{M_1 + M_2}} \tag{11}$$

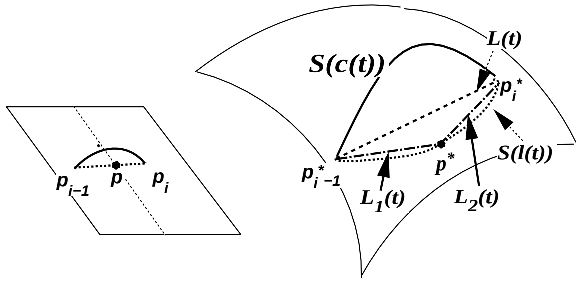

Fig. 2: The refinement of surface tessellation on trimming curve

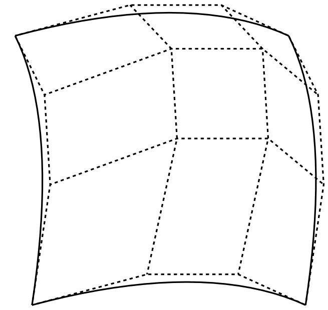

Fig. 3: The control polygon and the testing surface

The following paragraph will show that the proposed tessellation algorithm is robust against triangulation process of surface tessellation. Suppose $l(t)$ is the linear interpolant of $c(t)$ according to Δ_t in (11) and $l_\oplus(t)$ is the refinement of $l(t)$ by adding to $l(t)$ with the intersection points of $l(t)$ and surface tessellation lines. Then we show the chordal derivation between $L_{\bigoplus}(t)$, the lifted polygon of $l_\oplus(t)$, and $\Gamma(t)$ is still less than $\epsilon > 0$.

Let $I_i = [t_{i-1}, t_i]$ where $t_i = t_{i-1} + \Delta_t$, $P_{i-1} = c(t_{i-1})$ and $P_i = c(t_i)$. Without loss of generality, we assume the line segment $\overline{P_{i-1}P_i}$ is refined by a single point P after superimposing the surface tessellation lines. (The following explanation can be extended easily to the case with multiple refinement points.) Then there exists an $\eta_i \in I_i$ such that $\Gamma(t)$ is approximated by $L_1(t) = \overline{P_{i-1}^* P^*}, t \in [t_{i-1}, \eta_i]$ and $L_2(t) = \overline{P^* P_i^*}, t \in [\eta_i, t_i]$, where $P_{i-1}^* = S(P_{i-1}), P_i^* = S(P_i)$ and $P^* = S(P)$. (see Figure 2)

The chordal derivation between $\Gamma(t)$ and $L_\oplus(t)$ can be estimated as follows.

$$
\begin{aligned}
&|\Gamma(t) - L_\oplus(t)|_{t\in[t_{i-1},t_i]} \\
&= \max(|\Gamma(t) - L_1(t)|_{t\in[t_{i-1},\eta_i]},\ \ |\Gamma(t) - L_2(t)|_{t\in[\eta_i,t_i]}) \quad (12)
\end{aligned}
$$

Again by triangle inequality,

$$
\begin{aligned}
&|\Gamma(t) - L_\oplus(t)|_{t\in[t_{i-1},\eta_i]} \\
&\le |\Gamma(t) - S(l(t))| + |S(l(t)) - L_1(t)|, \ \ t \in [t_{i-1}, \eta_i] \\
&\le \max_{t\in[t_{i-1},t_i]} |\Gamma(t) - S(l(t))| + |S(l(t)) - L_1(t)|_{t\in[t_{i-1},\eta_i]}. \quad (13)
\end{aligned}
$$

According to (7) and the choice of the Δ_t in (10), we have $\max_{t\in[t_{i-1},t_i]} |\Gamma(t) - S(l(t))| \le \frac{M_1}{M_1+M_2}\epsilon$, where M_1 and M_2 are defined in (9). Moreover, since the step length $\eta_i - t_{i-1}$ is less than $\Delta_t = t_i - t_{i-1}$, the estimation of (8) is hold. That is, $|S(l(t)) - L(t)|_{t\in[t_{i-1},\eta_i]} \le \frac{M_2}{M_1+M_2}\epsilon$. Substitute these estimations back to (13), we have $|\Gamma(t) - L_1(t)|_{t\in[t_{i-1},\eta_i]} \le \epsilon$. Similarly $|\Gamma(t) - L_2(t)|_{t\in[\eta_i,t_i]} \le \epsilon$. From (12), we have $|\Gamma(t) - L_\oplus| \le \epsilon$, where $t \in [t_{i-1}, t_i]$.

4 Counter Examples and Experimental Results

We have developed a prototype system for modeling trimmed surface interactively. The system provides users to define surfaces and trimming curves, to view the trimmed surfaces, and to report numerical results. The system are implemented using Visual C++ on a desktop personal computer. In this system, two trimming curve tessellation algorithms are implemented. The counter examples discussed in Section 2.2 is constructed with the help of this interactive system.

The numerical tests given in Table 1 and Table 2 are based on the same Bézier surface with following 16 control points (Figure 3). $\{(-1,-1,-1), (-\frac{1}{3},-1,-\frac{1}{5}), (\frac{1}{3},-1,-\frac{1}{5})$,$(1,-1,-1)$, $(-1,-\frac{1}{3},-\frac{1}{5})$, $(-\frac{1}{3},-\frac{1}{3},1)$, $(\frac{1}{3},-\frac{1}{3},1)$, $(1,-\frac{1}{3},-\frac{1}{5})$, $(-1,\frac{1}{3},-\frac{1}{5})$, $(-\frac{1}{3},\frac{1}{3},1)$, $(\frac{1}{3},\frac{1}{3},1)$, $(1,\frac{1}{3},-\frac{1}{5})$, $(-1,1,-1)$, $(-\frac{1}{3},1,-\frac{1}{5})$, $(\frac{1}{3},1,-\frac{1}{5})$, $(1,1,-1)\}$. Furthermore, we use Bézier curves with control polygons $\{\ (.35,0.)\ $,$(.8,.3)\ $,$(.75,.9)$,$(.3,1.)\}$ and $\{(.35,0.)$, $(.95,.06)$, $(.75,.9)$, $(.32,1.)\}$ as trimming curves A and B respectively.

Figure 4 shows the triangulation behavior of trimming curve B in **D**. In addition to counter examples described in Section 2.2, there are two empirical tests are also reported in Table 1 and Table 2. The derivation errors of approximations for all three examples are listed in the columns $error_{L(t)}$ and $error_{L_\oplus(t)}$. The results indicate that the proposed algorithms do control derivation errors in 3D space.

Approx. method	$error_{l(t)}$	$error_{L(t)}$	$error_{L_\oplus(t)}$
Tessellation in **D**	.008295	.012213	.010278
Triangle Inequality	.000956	.001362	.001333

Table 1: Numerical results for trimming curve A as ϵ=.01

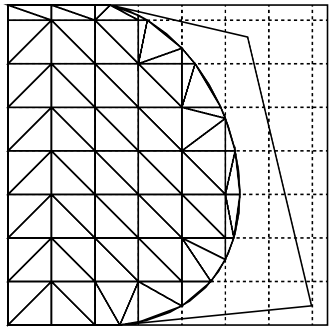

Fig. 4: Triangulation for trimmed surface in **D**

Approx. method	$error_{l(t)}$	$error_{L(t)}$	$error_{L_{\oplus}(t)}$
Tessellation in **D**	.066100	.066341	.091997
Triangle Inequality	.008141	.008488	.008997

Table 2: Numerical results for trimming curve B as ϵ=.087

5 Conclusion

This study addresses the problem in the linear tessellation of trimming curves. Counter examples are presented to show that existing trimmed surface tessellation algorithms do not assure the derivation error between 3D trimming curve and its linear tessellants. In order to remedy this flaw, we present a novel step length estimation method so that the trimming curve tessellation based on proposed step length always yield valid 3D approximation. The basic notion of our methods is to control the derivation error of curve approximation in 3D modeling space instead of in 2D parametric space. Besides, some empirical examples are given to demonstrate that our step length estimation result in correct approximation of 3D trimming curve.

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