Cognitive Conflicts as a Teaching Strategy to Enhance the Need of Mathematical Proofs with Technology Support

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ABSTRACT

In solving mathematical problems, students can feel that the universality of a conjecture or a formula is validated by their experiments and experiences. In contrast, students generally do not feel that deductive explanations strengthen their conviction that a conjecture or a formula is true. In order to cope with students' conviction based on empirical experience only and to create a need for deductive explanations, we developed a teaching activity with technology support intended to cause cognitive conflicts. In this paper, we describe the conducting process throughout this activity that led students to contradictions between original conjectures and new findings. The teacher could create familiar problem situations and use students' naïve inductive approaches to make students think mathematically and establish the necessity for proofs via technology support.

1: INTRODUCTION

For mathematicians, proofs have been considered as tools for verifying mathematical statements and explaining the reasons that support these statements [1]. Leibniz believed that a mathematical proof is a universal symbolic script, which allows one to distinguish clearly between truth and falsity [2]. Through logic and inductive reasoning, proofs provide students with other learning opportunities to enhance their mathematical understanding from a rigorous perspective. Therefore teaching proofs is a common activity in mathematics classroom of high school students, which is unique and different from other sciences teaching.

Proving is a complex and difficult task for high school students and the attempts to teach it were generally not successful in the past [3]. Euclidean geometry was just used as such a vehicle to teach formal mathematical proofs in high schools, and a great deal of time was devoted to it. Many students derived little or even no benefit from the geometry courses, and they could not distinguish between empirical evidence and deductive proofs [4]. It was very common for students to stop at the stage in which they had found a formula, and they could not feel the necessity of producing proofs. Students constructed a proof just because the teacher asked them to do so [5]. Even worse, many people thought of proofs as a part of geometry rather than a general mathematical process [6].

Therefore, how to have students engage in proving processes and let them see any reason or feel any need for it is an important issue in mathematics education [7]. In this paper, a teaching activity that raises students' consciousness of cognitive conflicts between old conjectures and new findings through technology support will be described. This activity created the setting and atmosphere from which the contradictions arose and left the findings unresolved. The need to explain and prove the findings students explored by themselves thus emerged quite naturally [1].

2: INSTRUCTIONAL FRAMEWORK

In this paper, the teaching activity was divided into three stages: *introduction*, *exploration in groups*, and *reporting back* [8]. In the *introduction* stage, the teacher created the problem situation and had students read the problem. Then the teacher discussed words or phrases students may not understand or led whole-class discussion to focus on the importance of understanding the problem. The main goal of this stage is to illustrate the importance of reading carefully, and to focus on special vocabulary, important data, and clarification process. In the second stage, *exploration in groups*, groups of size of three or four tried to solve problems by interacting with each other. During this stage, the teacher moved from group to group providing hints as needed, observed and questioned students about where they were, provided problem extensions in the right time, and required students who had obtained a solution to answer the question. The primary intention of this stage is to diagnose strengths and weaknesses of students' problem-solving process, and help students overcome blockages they encounter during this stage. In the final stage, *reporting back*, a member of each group would report back to the whole class. It is important that this stage is more than just presenting answers. By providing solutions, which may vary from group to group, students' strategies of heuristics may increase and thus be used on subsequent problem solving tasks. With skillful teaching, a variety of ideas may be discussed, which can be linked in an effort to enhance mathematical understanding. One other advantage of this reporting back stage is that students sometimes learn more easily from each other than they do from the teacher. The most important objective of this stage is to integrate different strategies and solutions, demonstrate general applicability of problem solving strategies, and show how problem features may influence solving approach.

3: THE PASTURE PROBLEM

Many students arriving at university level still does not even realize that fitting a formula to a pattern is not the same thing as proving it. How to help students bridge the gap from the conjecture to a proof and make them feel the need of proving is an important issue in a well-designed mathematics curriculum. Cognitive conflicts can provide just the new medium we need for teaching proofs. For this to be successful, however, we need a bank of good examples. The rest of this paper is devoted to one such, and the example of teaching the pasture problem will be described and presented based on the above teaching framework.

The Pasture Problem: *A shepherd has a rectangular pasture with a length of 90 meters and a width of 60 meters. The shepherd wants to construct a cross street on the pasture. Here are five designs (see Fig. 1 to Fig. 5). Do the following five figures have the same leftover area of the pasture? If not, which one of them would have the maximum leftover area of the pasture?*

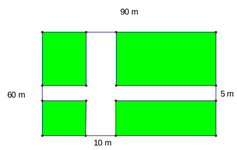

Fig. 1 Design 1

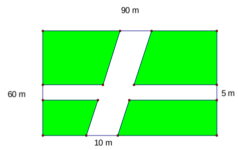

Fig. 2 Design 2

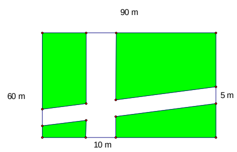

Fig. 3 Design 3

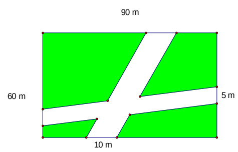

Fig. 4 Design 4

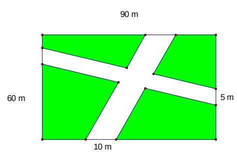

Fig. 5 Design 5

3.1: TEACHING STAGE 1: *INTRODUCTION*

The purpose of this teaching stage is to make students understand *the pasture problem*, including reading and rereading the problem, initial and subsequent representations of the problem, analysis of the information and conditions of the problem, and assessment of difficulty in the problem. The teacher has to create the problem situation and pose *the pasture problem* described above and then he or she must consider students' ability to identify the problem and define it. Students will code the important elements from the problem situation. They will represent the characteristics of the pasture problem mentally, involving relating the newly acquired information to the previously acquired information. Then the teacher gives every student a chance to guess the answer and judge the reason. Almost ninety five percent of students in the class would consider that the five figures all have the same leftover area, $440\,m^2$. This is because they think that the four leftover pastures could combine into a large rectangle with a length of (90-10) meters and a width of (60-5) meters. This conjecture students gained plays an important role during *introduction* stage because it will lead students to generate cognitive conflicts during the next teaching stage.

3.2: TEACHING STAGE 2: *EXPLORATION IN GROUPS*

The purpose of this teaching stage is to make students plan how to proceed and to execute the solution according to the plan, consisting of identifying goals and sub-goals, making and implementing a global plan, monitoring and controlling the progress of a solution plan. The teacher divided the class into eight groups. There were four students with heterogonous in the mathematical ability in each group. Then the teacher provided each group the computer tool [9] which could simulate the pasture problem, help students explore it, and guide them to form new conjectures. Students used this tool to investigate the nature of the pasture problem, and to monitor progress of their plan of the solution (Fig. 6, Fig. 7, and Fig.8 are the displays of operations of this computer tool). The teacher also had to move from group to group providing assistance via scaffolding. Through group discussion and technology support, almost each group of students found the following facts: (a) the four leftover pastures can combine into a big rectangle in the first three figures (see Fig. 6). (b) In the fourth figure, the four leftover pastures can combine into a big rectangle, but there is a small overlap of a parallelogram in the middle of the big rectangle (see Fig. 7). (c) In the final figure, the four leftover pastures can combine into a big rectangle, but there is a small gap of a parallelogram in the middle of the big rectangle (see Fig. 8).

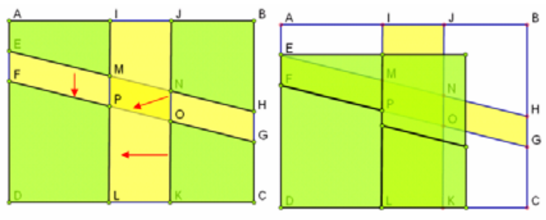

Fig. 6 The finding in condition 1

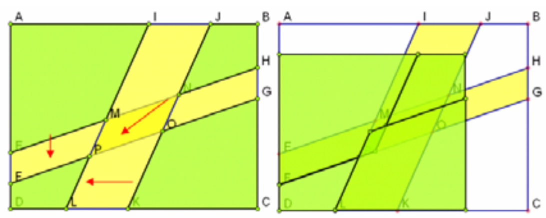

Fig. 7 The finding in condition 2

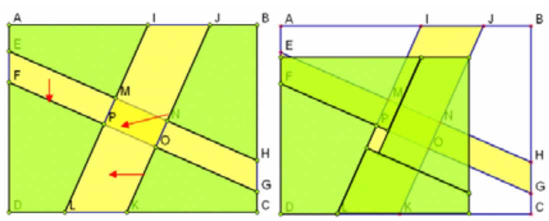

Fig. 8 The finding in condition 3

Fact (a) is an expected result whereas fact (b) and fact (c) are surprising findings. This is because students hypothesized that the four leftover pastures could combine into a big rectangle in all five figures and this intuitive belief was quite strong especially when the first three figures are checked using the computer tool. But the findings of the last two figures didn't support their judgment and original conjecture. Therefore they were much surprised about the strange phenomenon occurred in their exploration via computer support. The teacher should utilize the above three facts to guide students to resolve the contradictions. Cognitive conflicts resulting from these contradictions while checking their original conjectures might trigger a need for explanations and proofs. Students in the same group started to discuss why these surprising phenomena occur and they desired to build a mathematical model to address this issue.

3.3: TEACHING STAGE 3: *REPORTING BACK*

The purpose of this teaching stage is to make students evaluate what they know about their performance, encompassing the interaction of a person, a solution and a strategy. In this final stage, the representatives of each group will report back to the whole class. Each group would get the new conjecture of the contradictory phenomenon and everyone concerned not only the fact of this phenomenon but also the reason why this phenomenon occurred. The teacher would need to listen carefully to the reports of the members who represent their groups and discuss the key points of their solutions. After all the representatives have finished their reporting, the teacher needed to summarize different approaches to the pasture problem, eventually leading to a final solution that might be more elegant. However, only two of the eight groups in a class could build a model to solve this problem and explain the results of the contradictions and surprise. The following is the solution provided by one of the two successful groups in explaining the strange phenomenon.

Because the area of the two roads is always fixed according to the problem situation, we can get the sum of the area of the four leftover pastures by subtracting the area of the two roads, IJKL, EFGH from the area of rectangle ABCD, and then adding the area of the parallelogram MNOP, the intersection of the two roads (see Fig. 9). Therefore the larger the area of MNOP is; the greater the sum of the area of the four leftover pastures is. We define that the width of the vertical road is x (i.e. $\overline{IJ}=x$*), and the width of the horizontal road is y (i.e.* $\overline{EF}=y$*). It is supposed that the included angle of the two roads is θ (i.e.* $\angle MPO=\theta$*), and the included angle of the vertical road IJKL and* $\overline{AD}$ *is α (i.e.* $\angle PMR=\alpha$*). We also construct that* $\overline{OQ}$ *is perpendicular to* $\overline{MP}$*,* $\overline{MR}$ *is* *parallel to* $\overline{AD}$*, and* $\overline{OS}$ *is parallel to* $\overline{AB}$*. Observing △MPR, we can find that*

$$\frac{y}{\sin\theta}=\frac{\overline{MP}}{\sin(180^\circ-\alpha-\theta)}$$

$$\Rightarrow \overline{MP}=\frac{y\times\sin(\alpha+\theta)}{\sin\theta}$$

Similarly observing △ QSO, we can get that $\frac{x}{\sin 90^\circ}=\frac{\overline{OQ}}{\sin(90^\circ-\alpha)}\Rightarrow \overline{OQ}=x\times\cos\alpha$ *. Hence the area of the parallelogram MNOP is equal to*

$$x\times y\times\cos\alpha\times\frac{\sin(\alpha+\theta)}{\sin\theta}$$

$$=x\times y\times\cos\alpha\times\frac{\sin\alpha\cos\theta+\cos\alpha\sin\theta}{\sin\theta}$$

$$=x\times y\times\cos\alpha\times(\sin\alpha\cot\theta+\cos\alpha)$$

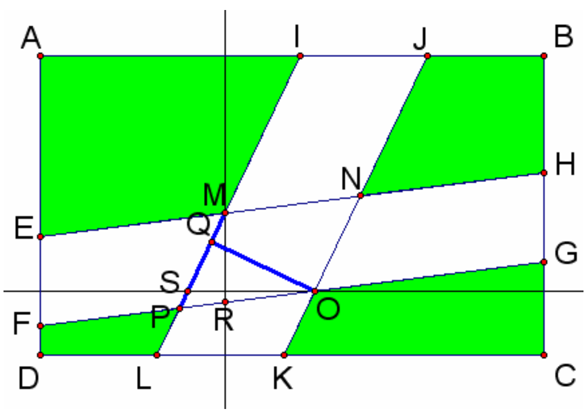

Fig. 9 The draft of the formal proof

From this formula, we could consider the following conditions of the pasture problem.

(1) When the included angle α of vertical road and $\overline{AD}$ *is fixed, we could find that the larger the included angle θ is, the smaller the area of parallelogram MNOP becomes. This is because when θ increases,* $\cot\theta$ *decreases.*

(2) When the vertical road EFGH is parallel to $\overline{AD}$ *(i.e. α=0°), the area of parallelogram MNOP is equal to* $\overline{IJ}\times\overline{EF}$ *(i.e. x × y).*

(3) When the horizontal road IJKL is parallel to $\overline{AB}$ *(i.e.* $\alpha+\theta=90^\circ$*), the area of parallelogram MNOP is equal to* $\overline{IJ}\times\overline{EF}$ *(i.e. x × y) on account of*

$$\cos\alpha\times\frac{\sin(\alpha+\theta)}{\sin\theta}=\cos(90^\circ-\theta)\times\frac{\sin 90^\circ}{\sin\theta}$$

$$=\sin\theta\times\frac{1}{\sin\theta}=1$$

4: Conclusion

In this paper, we exemplified a teaching design which students encountered cognitive conflicts with computer support and had the opportunities for feeling the need to prove, rather than considering proving as unnecessary. Cognitive conflicts occur when expectations are not fulfilled. Our task for *the pasture problem* takes cognitive conflicts as a teaching strategy to encourage students to explore more and to bridge the gap between a conjecture and a proof. In the beginning, students made a conjecture concerning the solution of *the pasture problem* and found a reason why it was true. The reason was often rooted in common sense or based on previous learning. Through computer-supported cognitive conflicts, our teaching activity led students to more than accepting the correctness of the new conjecture; it led them to construct a new explanation for this new conjecture naturally. It is believed that in this task students were guided to use deductive reasoning to construct reasons to support the new conjecture that motivated them to solve the pasture problem.

Technology makes students engage in exploration more easily in this activity and enable them to try various possibilities of solving the problem. Like a jungle adventure, students explore uncertainties and encounter different new things during this process. They discover that something is not as predicted as they originally considered. Consequently students will think about how to explain this phenomenon and finally develop a better solution integrating the previous experiences. The design with technology support has brought proofs into the realm of student activity and argument; that is, proofs have been engaged naturally in true mathematical activities. And indeed in this task, students ceased to be recipients of formal proofs, but were engaged in an activity of construction and evaluation of conjectures where certainty and understanding were not clear, and they had to use their mathematical knowledge to explain contradictions and overcome uncertainties with computer support [1].

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