H_∞ Control for Time-Delay Systems Via LMI Optimization Approach

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Abstract

Robust H_∞ control for a class of linear time-delay systems is considered. Improved delay-dependent H_∞ control criteria are proposed to minimize the H_∞-norm bound via LMI optimization approach. Based on the result of this paper, model transformation and bounded techniques on cross product terms are not used in finding the delay-dependent results. Linear matrix inequality (LMI) optimization approach is used to design the robust H_∞ state feedback control. Some numerical examples are given to illustrate the effectiveness of the main results.

1. Iutroduction

It is well known that the existence of the delay in a dynamic system may cause instability or bad system performances in open and closed-loop systems [5]. In many practical systems, time delay is often encountered in various systems, such as chemical engineering systems, distributed networks, inferred grinding model, manual control, microwave oscillator, neural network, population dynamic model, ship stabilization, and systems with lossless transmission lines. Furthermore the system model always contains some uncertain elements; these uncertainties may be due to additive unknown noise, environmental influence, poor plant knowledge [7]. Hence the robust control is developed to stabilize the uncertain time-delay systems; see for example, [2-5, 8, 10-11].

In the recent year, the H_∞ control problem for time-delay systems has been an active topic in control system theory [2-4, 10-11]. The H_∞ control was proposed to reduce the effect of the disturbance input on the regulated output to within a prescribed level. Riccati-equation-based approach was proposed in [2, 4, 11] for H_∞ control, but this approach is not easy to find the minimal H_∞-norm bound (γ) and the suitable controller. In [3], the LMI approach had been used to design the H_∞ control for a given H_∞-norm bound (γ). In [10], the delay-dependent H_∞ control criteria were proposed by using the Park inequality [9]. The bounded inequality technique [9] will caused some conservatism and the LMI optimization results in [10] will cause the high state feedback gains; see the Example of [10]. In the past, some model transformations are used to obtain the delay-dependent stability criteria, but the model transformation techniques will also cause conservatism for the stability analysis. In this paper, the H_∞ control is developed without using model transformation and bounded inequality technique on related cross product terms. LMI optimization approach and numerical searching algorithm will be used to find the minimization of H_∞-norm bound. Some numerical examples are given to illustrate the use of the results.

Notation. For a matrix A, we denote the standard Euclidean norm by $\|A\|$, the transpose by A^T, rank by $rank(A)$, minimal eigenvalue by $\lambda_{\min}(A)$, maximal eigenvalue by $\lambda_{\max}(A)$, and symmetric positive (nagative) definite by $A>0$ ($A<0$). I means identity matrix. $A \le B$ means that matrix $B - A$ is symmetric positive semi-definite. $\|f\|_2 = \sqrt{\int_0^\infty \|f(t)\|^2 dt}$, $f(t) \in L_2[0,\infty)$. $\|f(t)\|$ means the Euclidean vector norm at time t. $L_2[0,\infty)$ stands for space of square integrable functions on $[0,\infty)$. C_0 means that the set of all continuous functions from $[-h,0]$ to $\Re^n$.

2. Problem formulation and main results

Consider the following time-delay system:

$$\dot{x}(t) = A_1 x(t) + A_2 x(t-h) + B_1 u(t) + B_2 w(t), \tag{1a}$$

$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{1b}$$

$$z(t) = Cx(t) + Du(t), \tag{1c}$$

where $x \in \Re^n$, x_t is the state at time t defined by $x_t(\theta) := x(t+\theta),\ \forall\theta \in [-h,0]$, $u \in \Re^m$ is the input, $w \in \Re^l$ is the disturbance input, $z \in \Re^q$ is the regulated output, $A_1 \in \Re^{n\times n}$, $A_2 \in \Re^{n\times n}$, $B_1 \in \Re^{n\times m}$, $B_2 \in \Re^{n\times l}$, $C \in \Re^{q\times n}$, and $D \in \Re^{q\times m}$ are constant matrices, $h > 0$ is the time delay, $\phi \in C_0$ is the initial valued function.

Definition 1. [11]

Consider the system (1) with $u(t) = -Kx(t)$ and the following conditions are satisfied:

(i) With $w(t) = 0$, the closed-loop system (1) with $u(t) = -Kx(t)$ is asymptotically stable.

(ii) With zero initial condition (i.e. $\phi = 0$), the signals $w(t)$ and $z(t)$ are bounded by

$\int_0^\infty \|z(t)\|^2 dt \le \gamma^2 \cdot \int_0^\infty \|w(t)\|^2 dt$, (i.e., $\|z\|_2^2 \le \gamma^2 \cdot \|w\|_2^2$)
$\forall\ w \in L_2[0,\infty),\ w \neq 0$,
for a constant $\gamma > 0$. In this condition, the system (1) is said to be stabilizable with disturbance attenuation γ, and the control law $u(t) = -Kx(t)$ is said to be an H_∞ control for system (1). The parameter γ is said to be the H_∞-norm bound for the H_∞ state feedback control.

Systems (1) with $u(t) = -Kx(t)$ can be rewritten as

$$\dot{x}(t) = (A_1 - B_1 K)x(t) + A_2 x(t-h) + B_2 w(t),$$
$$z(t) = (C - DK)x(t). \qquad (2)$$

For a given controller gain $K \in \Re^{m\times n}$, the H_∞-norm bound can be solved from the following result.

Theorem 1. Consider the system (1) with $u(t) = -Kx(t)$. Suppose the following optimization problem:

$$\min_{\bar{\gamma}, P_1, P_2, P_3, Q_1, Q_2, Q_3} \bar{\gamma}, \qquad (3a)$$

subject to the following LMI:

$$\Omega_0 = \begin{bmatrix} \Pi_{11} & P_1 A_2 + Q_1 - Q_2^T & h\cdot Q_1 & P_1 B_2 - Q_3^T & h\cdot (A_1 - B_1 K)^T P_3 \\ A_2^T P_1 + Q_1^T - Q_2 & -P_2 + Q_2 + Q_2^T & h\cdot Q_2 & Q_3^T & h\cdot A_2^T P_3 \\ h\cdot Q_1^T & h\cdot Q_2^T & -h\cdot P_3 & h\cdot Q_3^T & 0 \\ B_2^T P_1 - Q_3 & Q_3 & h\cdot Q_3 & -\bar{\gamma}\cdot I & h\cdot B_2^T P_3 \\ h\cdot P_3(A_1 - B_1 K) & h\cdot P_3 A_2 & 0 & h\cdot P_3 B_2 & -h\cdot P_3 \end{bmatrix} < 0, \quad (3b)$$

has a solution $\bar{\gamma} > 0$, matrices $P_1 \in \Re^{n\times n} > 0$, $P_2 \in \Re^{n\times n} > 0$, $P_3 \in \Re^{n\times n} > 0$, $Q_1 \in \Re^{n\times n}$, $Q_2 \in \Re^{n\times n}$, and $Q_3 \in \Re^{l\times n}$, where

$\Pi_{11} = (A_1 - B_1 K)^T P_1 + P_1(A_1 - B_1 K) + P_2 - Q_1 - Q_1^T + (C - DK)^T (C - DK)$.

Then the system (1) is stabilizable by H_∞ control $u(t) = -Kx(t)$ with disturbance attenuation $\gamma = \sqrt{\bar{\gamma}}$.

Proof.

Define the Lyapunov function as

$$V(x_t) = x^T(t) P_1 x(t) + \int_{t-h}^{t} x^T(s) P_2 x(s) ds + \int_{-h}^{0}\int_{t+s}^{t} \eta^T(x_\tau) P_3 \eta(x_\tau) d\tau ds, \qquad (4)$$

where $P_i = \bar{P}_i^{-1} > 0$, $i = 1, 2$, $\eta(x_t) = \dot{x}(t) = (A_1 - B_1 K)x(t) + A_2 x(t-h) + B_2 w(t)$. By the system (2), we have

$$\int_{t-h}^{t} \eta(x_s) ds = \int_{t-h}^{t} \dot{x}(s) ds = x(t) - x(t-h).$$

The time derivative of $V(x_t)$ in (4), along the trajectories of (2) is given by

$$\begin{aligned}\dot{V}(x_t) &= \dot{x}^T(t) P_1 x(t) + x^T(t) P_1 \dot{x}(t) + x^T(t) P_2 x(t) \\ &- x^T(t-h) P_2 x(t-h) + h\cdot \eta^T(x_t) P_3 \eta(x_t) \\ &- \int_{t-h}^{t} \eta^T(x_s) P_3 \eta(x_s) ds \\ &= [(A_1 - B_1 K)x(t) + A_2 x(t-h) + B_2 w(t)]^T P_1 x(t) \\ &+ x^T(t) P_1 [(A_1 - B_1 K)x(t) + A_2 x(t-h) + B_2 w(t)] \\ &+ x^T(t) P_2 x(t) - x^T(t-h) P_2 x(t-h) \\ &+ h\cdot [(A_1 - B_1 K)x(t) + A_2 x(t-h) + B_2 w(t)]^T P_3 \\ &\cdot [(A_1 - B_1 K)x(t) + A_2 x(t-h) + B_2 w(t)] \\ &- \int_{t-h}^{t} \eta^T(x_s) P_3 \eta(x_s) ds \\ &+ 2x^T(t) Q_1 \cdot \left[\int_{t-h}^{t} \eta(x_s) ds - x(t) + x(t-h)\right] \\ &+ 2x^T(t-h) Q_2 \left[\int_{t-h}^{t} \eta(x_s) ds - x(t) + x(t-h)\right] \\ &+ 2w^T(t) Q_3 \left[\int_{t-h}^{t} \eta(x_s) ds - x(t) + x(t-h)\right],\end{aligned}$$

where Q_1, Q_2, and Q_3, are some matrices. Define a function by

$\hat{J}(x(t), w(t)) = \dot{V}(x_t) + z^T(t) z(t) - \bar{\gamma}\cdot w^T(t) w(t)$, (5a)

where $\bar{\gamma} = \gamma^2$. Note that $z(t) = (C - DK)x(t)$, we have

$$\begin{aligned}\hat{J}(x(t), w(t)) &= \dot{V}(x_t) + x^T(t)(C - DK)^T (C - DK)x(t) \\ &\quad - \bar{\gamma}\cdot w^T(t) w(t) \\ &= \frac{1}{h}\int_{t-h}^{t} \begin{bmatrix} x(t) \\ x(t-h) \\ \eta(x_s) \\ w(t) \end{bmatrix}^T \cdot \Omega_1 \cdot \begin{bmatrix} x(t) \\ x(t-h) \\ \eta(x_s) \\ w(t) \end{bmatrix} ds, \end{aligned} \qquad (5b)$$

where

$$\Omega_1 = \begin{bmatrix} \Pi_{11} & P_1 A_2 + Q_1 - Q_2^T & h\cdot Q_1 & P_1 B_2 - Q_3^T \\ A_2^T P_1 + Q_1^T - Q_2 & -P_2 + Q_2 + Q_2^T & h\cdot Q_2 & Q_3^T \\ h\cdot Q_1^T & h\cdot Q_2^T & -h\cdot P_3 & h\cdot Q_3^T \\ B_2^T P_1 - Q_3 & Q_3 & h\cdot Q_3 & -\bar{\gamma}\cdot I \end{bmatrix} + \begin{bmatrix} h\cdot (A_1 - B_1 K)^T P_3 \\ h\cdot A_2^T P_3 \\ 0 \\ h\cdot B_2^T P_3 \end{bmatrix} (h\cdot P_3)^{-1} [h\cdot P_3(A_1 - B_1 K) \quad h\cdot P_3 A_2 \quad 0 \quad h\cdot P_3 B_2].$$

By the Schur complement of [1] with matrix Ω_0 in (3b), we have

$$\Omega_1 < 0. \qquad (5c)$$

From (5b) and (5c) with $w(t) = 0$, there exists a constant $\alpha > 0$ such that

$$\dot{V}(x_t)\big|_{w(t)=0} \le -\alpha \|x(t)\|^2.$$

Hence the closed system (1) with $u(t)=-Kx(t)$ and $w(t)=0$ is asymptotically stable [5]. Integrating the function in (5a) from 0 to ∞ and by (5b)-(5c), we have

$$V(x_\infty)-V(\phi)+\|z\|_2^2-\gamma^2\cdot\|w\|_2^2\le 0 .$$

With zero initial condition ($\phi=0$), we have

$$V(\phi)=0,\ V(x_\infty)\ge 0,$$

and

$$\|z\|_2^2\le\gamma^2\cdot\|w\|_2^2,\ \forall\, w\in L_2[0,\infty),\, w\ne 0 .$$

By the Definition 1, the system (1) is stabilizable by H_∞ control $u(t)=-Kx(t)$ with disturbance attenuation $\gamma=\sqrt{\bar\gamma}$.

In the following, we will solve the controller gain K from the following LMI result.

Corollary 1.

Suppose the following optimization problem:

$$\min_{\bar\gamma,\bar P_1,\hat P_2,\hat K,\hat Q_1,\hat Q_2,\hat Q_3}\bar\gamma, \qquad (6a)$$

subject to the follwing LMI:

$$\begin{bmatrix} \hat\Pi_{11} & \Pi_{12} & h\cdot\hat Q_1 & B_2-\hat Q_3^T & \Pi_{15} & \Pi_{16}\\ \Pi_{12}^T & \Pi_{22} & h\cdot\hat Q_2 & \hat Q_3^T & h\cdot\bar P_1A_2^T & 0\\ h\cdot\hat Q_1^T & h\cdot\hat Q_2^T & -h\cdot\bar P_1 & h\cdot\hat Q_3^T & 0 & 0\\ B_2^T-\hat Q_3 & \hat Q_3 & h\cdot\hat Q_3 & -\bar\gamma\cdot I & h\cdot B_2^T & 0\\ \Pi_{15}^T & h\cdot A_2\bar P_1 & 0 & h\cdot B_2 & -h\cdot\bar P_1 & 0\\ \Pi_{16}^T & 0 & 0 & 0 & 0 & -I \end{bmatrix}<0, \qquad (6b)$$

has a solution $\bar\gamma>0$, $\bar P_1\in\Re^{n\times n}>0$, $\bar P_2\in\Re^{n\times n}>0$, $\hat K\in\Re^{m\times n}$, $\hat Q_1\in\Re^{n\times n}$, $\hat Q_2\in\Re^{n\times n}$, and $\hat Q_3\in\Re^{l\times n}$, where

$$\hat\Pi_{11}=\bar P_1A_1^T+A_1\bar P_1-B_1\hat K-\hat K^TB_1^T+\hat P_2-\hat Q_1-\hat Q_1^T,$$

$$\Pi_{12}=A_2\bar P_1+\hat Q_1-\hat Q_2^T,\ \Pi_{22}=-\hat P_2+\hat Q_2+\hat Q_2^T,$$

$$\Pi_{15}=h\cdot\left(\bar P_1A_1^T-\hat K^TB_1^T\right),\ \Pi_{16}=\bar P_1C^T-\hat K^TD^T . \qquad (6c)$$

Then the system (1) is stabilizable by H_∞ control $u(t)=-Kx(t)=-\hat K\bar P_1^{-1}x(t)$ with disturbance attenuation $\gamma=\sqrt{\bar\gamma}$.

Proof.

In order to find the controller gain K from LMI, we choose $P_1=P_3$. Pre- and post-multiplying the matrix Ω_0 in (3b) by $diag[\bar P_1\ \ \bar P_1\ \ \bar P_1\ \ I\ \ \bar P_1]>0$, where $\bar P_1=P_1^{-1}$, we can define

$$\hat P_2=\bar P_1P_2\bar P_1,\ \hat Q_1=\bar P_1Q_1\bar P_1,\ \hat Q_2=\bar P_1Q_2\bar P_1,$$

$$\hat Q_3=Q_3\bar P_1,\ \hat K=K\bar P_1 .$$

By Schur complement of [1], the condition (6b) could be obtained from (3b).

In the next, we will consider the following uncertain time-delay system:

$$\dot x(t)=A_1(t)x(t)+A_2(t)x(t-h)+B_1(t)u(t)+B_2(t)w(t), \qquad (7a)$$

$$x(t)=\phi(t),\ t\in[-h,0], \qquad (7b)$$

$$z(t)=Cx(t)+Du(t), \qquad (7c)$$

where $A_1(t)=A_1+\Delta A_1(t)$, $A_2(t)=A_2+\Delta A_2(t)$, $B_1(t)=B_1+\Delta B_1(t)$, $B_2(t)=B_2+\Delta B_2(t)$, $\Delta A_1(t)$, $\Delta A_2(t)$, $\Delta B_1(t)$, and $\Delta B_2(t)$, are some perturbed matrices.

(A1) The perturbed matrices $\Delta A_1(t)$, $\Delta A_2(t)$, $\Delta B_1(t)$, and $\Delta B_2(t)$ satisfy

$$[\Delta A_1(t)\ \ \Delta A_2(t)\ \ \Delta B_1(t)\ \ \Delta B_2(t)]=M\cdot F(t)\cdot[N_1\ \ N_2\ \ N_3\ \ N_4],$$

where $M\in\Re^{n\times\mu}$, $N_1\in\Re^{\mu\times n}$, $N_2\in\Re^{\mu\times n}$, $N_3\in\Re^{\mu\times m}$, and $N_4\in\Re^{\mu\times l}$ are some given constant matrices and $F(t)\in\Re^{\mu\times\mu}$ satisfies

$$F^T(t)F(t)\le I .$$

For a given controller gain $K\in\Re^{m\times n}$, the H_∞-norm bound could be solved from the following result.

Theorem 2.

Consider the system (7) and (A1) with $u(t)=-Kx(t)$. Suppose the following optimization problem:

$$\min_{\bar\gamma,\varepsilon,P_1,P_2,P_3,Q_1,Q_2,Q_3}\bar\gamma, \qquad (8a)$$

subject to the following LMI:

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & h\cdot Q_1 & \Sigma_{14} & h\cdot(A_1-B_1K)^TP_3 & P_1M\\ \Sigma_{12}^T & \Sigma_{22} & h\cdot Q_2 & Q_3^T+\varepsilon\cdot N_2^TN_4 & h\cdot A_2^TP_3 & 0\\ h\cdot Q_1^T & h\cdot Q_2^T & -h\cdot P_3 & h\cdot Q_3^T & 0 & 0\\ \Sigma_{14}^T & Q_3+\varepsilon\cdot N_4^TN_2 & h\cdot Q_3 & -\bar\gamma\cdot I+\varepsilon\cdot N_4^TN_4 & h\cdot B_2^TP_3 & 0\\ h\cdot P_3(A_1-B_1K) & h\cdot P_3A_2 & 0 & h\cdot P_3B_2 & -h\cdot P_3 & h\cdot P_3M\\ M^TP_1 & 0 & 0 & 0 & h\cdot M^TP_3 & -\varepsilon\cdot I \end{bmatrix}<0, \qquad (8b)$$

has a solution $\bar\gamma>0$, $\varepsilon>0$, $P_1\in\Re^{n\times n}>0$, $P_2\in\Re^{n\times n}>0$, $P_3\in\Re^{n\times n}>0$, $Q_1\in\Re^{n\times n}$, $Q_2\in\Re^{n\times n}$, and $Q_3\in\Re^{l\times n}$, where

$$\Sigma_{11}=(A_1-B_1K)^TP_1+P_1(A_1-B_1K)+P_2-Q_1-Q_1^T+(C-DK)^T(C-DK)+\varepsilon\cdot(N_1-N_3K)^T(N_1-N_3K),$$

$$\Sigma_{12}=P_1A_2+Q_1-Q_2^T+\varepsilon\cdot(N_1-N_3K)^TN_2,$$

$$\Sigma_{14}=P_1B_2-Q_3^T+\varepsilon\cdot(N_1-N_3K)^TN_4,$$

$$\Sigma_{22}=-P_2+Q_2+Q_2^T+\varepsilon\cdot N_2^TN_2 .$$

Then the system (7) with (A1) is stabilizable by H_∞ control $u(t)=-Kx(t)$ with disturbance attenuation

$\gamma = \sqrt{\bar{\gamma}}$.

Proof.

Redefine the function $\eta(x_t)$ in the proof of Theorem 1 by

$\eta(x_t) = [A_1(t) - B_1(t)K]x(t) + A_2(t)x(t-h) + B_2(t)w(t)$.

By the same technique of Theorem 1 with (3b), we have

$$\begin{bmatrix} \tilde{\Pi}_{11} & P_1A_2(t)+Q_1-Q_2^T & h\cdot Q_1 & P_1B_2(t)-Q_3^T & h\cdot(A_1(t)-B_1(t)K)^T P_3 \\ A_2^T(t)P_1+Q_1^T-Q_2 & -P_2+Q_2+Q_2^T & h\cdot Q_2 & Q_3^T & h\cdot A_2^T(t)P_3 \\ h\cdot Q_1^T & h\cdot Q_2^T & -h\cdot P_3 & h\cdot Q_3^T & 0 \\ B_2^T(t)P_1-Q_3 & Q_3 & h\cdot Q_3 & -\bar{\gamma}\cdot I & h\cdot B_2^T(t)P_3 \\ h\cdot P_3(A_1(t)-B_1(t)K) & h\cdot P_3A_2(t) & 0 & h\cdot P_3B_2(t) & -h\cdot P_3 \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_{11} & P_1A_2+Q_1-Q_2^T & h\cdot Q_1 & P_1B_2-Q_3^T & h\cdot(A_1-B_1K)^T P_3 \\ A_2^TP_1+Q_1^T-Q_2 & -P_2+Q_2+Q_2^T & h\cdot Q_2 & Q_3^T & h\cdot A_2^TP_3 \\ h\cdot Q_1^T & h\cdot Q_2^T & -h\cdot P_3 & h\cdot Q_3^T & 0 \\ B_2^TP_1-Q_3 & Q_3 & h\cdot Q_3 & -\bar{\gamma}\cdot I & h\cdot B_2^TP_3 \\ h\cdot P_3(A_1-B_1K) & h\cdot P_3A_2 & 0 & h\cdot P_3B_2 & -h\cdot P_3 \end{bmatrix} + \Gamma F(t)\Lambda^T + \Lambda F^T(t)\Gamma^T ,$$

where $\tilde{\Pi}_{11} = [A_1(t) - B_1(t)K]^T P_1 + P_1[A_1(t) - B_1(t)K]$
$+ P_2 - Q_1 - Q_1^T + (C - DK)^T(C - DK)$,

$\Gamma = \begin{bmatrix} M^TP_1 & 0 & 0 & 0 & h\cdot M^TP_3 \end{bmatrix}^T$,

$\Lambda = \begin{bmatrix} N_1 - N_3K & N_2 & 0 & N_4 & 0 \end{bmatrix}^T$.

Since $\Gamma F(t)\Lambda^T + \Lambda F^T(t)\Gamma^T \le \varepsilon^{-1}\cdot\Gamma\Gamma^T + \varepsilon\cdot\Lambda\Lambda^T$, $\varepsilon > 0$, and by Schur complement with (8b), we can complete this proof.

In the following, we will solve the controller gain K from the following LMI result.

Corollary 2.

Suppose the following optimization problem:

$$\min_{\bar{\gamma},\varepsilon,\bar{P}_1,\bar{P}_2,\hat{K},\hat{Q}_1,\hat{Q}_2,\hat{Q}_3} \bar{\gamma} , \qquad (9a)$$

subject to the follwing LMI:

$$\begin{bmatrix} \hat{\hat{\Pi}}_{11} & \Pi_{12} & h\cdot\hat{Q}_1 & B_2-\hat{Q}_3^T & \hat{\hat{\Pi}}_{15} & \Pi_{16} & \bar{P}_1N_1^T-\hat{K}^TN_3^T \\ \Pi_{12}^T & \Pi_{22} & h\cdot\hat{Q}_2 & \hat{Q}_3^T & h\cdot\bar{P}_1A_2^T & 0 & \bar{P}_1N_2^T \\ h\cdot\hat{Q}_1^T & h\cdot\hat{Q}_2^T & -h\cdot\bar{P}_1 & h\cdot\hat{Q}_3^T & 0 & 0 & 0 \\ B_2^T-\hat{Q}_3 & \hat{Q}_3 & h\cdot\hat{Q}_3 & -\bar{\gamma}\cdot I & h\cdot B_2^T & 0 & N_4^T \\ \hat{\hat{\Pi}}_{15}^T & h\cdot A_2\bar{P}_1 & 0 & h\cdot B_2 & -h\cdot\bar{P}_1+\varepsilon\cdot h^2\cdot MM^T & 0 & 0 \\ \Pi_{16}^T & 0 & 0 & 0 & 0 & -I & 0 \\ N_1\bar{P}_1-N_3\hat{K} & N_2\bar{P}_1 & 0 & N_4 & 0 & 0 & -\varepsilon\cdot I \end{bmatrix} < 0 \text{ ,(9b)}$$

has a solution $\bar{\gamma} > 0$, $\varepsilon > 0$, $\bar{P}_1 \in \Re^{n\times n} > 0$, $\bar{P}_2 \in \Re^{n\times n} > 0$, $\hat{K} \in \Re^{m\times n}$, $\hat{Q}_1 \in \Re^{n\times n}$, $\hat{Q}_2 \in \Re^{n\times n}$, and $\hat{Q}_3 \in \Re^{l\times n}$, where

$\hat{\hat{\Pi}}_{11} = \bar{P}_1A_1^T + A_1P_1 - B_1\hat{K} - \hat{K}^TB_1^T + \hat{P}_2 - \hat{Q}_1 - \hat{Q}_1^T + \varepsilon\cdot MM^T$,

$\hat{\hat{\Pi}}_{15} = h\cdot\left(\bar{P}_1A_1^T - \hat{K}^TB_1^T + \varepsilon\cdot MM^T\right)$.

Then the system (7) with (A1) is stabilizable by H_∞ control $u(t) = -Kx(t) = -\hat{K}\bar{P}_1^{-1}x(t)$ with disturbance attenuation $\gamma = \sqrt{\bar{\gamma}}$.

Proof.

By the proof of Corollary 1 and the fact $\Gamma F(t)\Lambda^T + \Lambda F^T(t)\Gamma^T \le \varepsilon\cdot\Gamma\Gamma^T + \varepsilon^{-1}\cdot\Lambda\Lambda^T$, $\varepsilon > 0$, we can prove the results in the similar way of Theorem 2

In the following, we can obtain a stability criterion from Theorem 2 with $u(t) = w(t) = B_1(t) = B_2(t) = 0$ of system (7a).

Corollary 3: The system (7a) with (A1) and $u(t) = w(t) = B_1(t) = B_2(t) = 0$ is asymptotically stable, if there exist a scalar $\varepsilon > 0$, matrices $P_1 \in \Re^{n\times n} > 0$, $P_2 \in \Re^{n\times n} > 0$, $P_3 \in \Re^{n\times n} > 0$, $Q_1 \in \Re^{n\times n}$, $Q_2 \in \Re^{n\times n}$, and $Q_3 \in \Re^{l\times n}$, such that the following LMI holds

$$\begin{bmatrix} \tilde{\tilde{\Pi}}_{11} & P_1A_2+Q_1-Q_2^T+\varepsilon\cdot N_1^TN_2 & h\cdot Q_1 & h\cdot A_1^TP_3 & P_1M \\ A_2^TP_1+Q_1^T-Q_2+\varepsilon\cdot N_2^TN_1 & -P_2+Q_2+Q_2^T+\varepsilon\cdot N_2^TN_2 & h\cdot Q_2 & h\cdot A_2^TP_3 & 0 \\ h\cdot Q_1^T & h\cdot Q_2^T & -h\cdot P_3 & 0 & 0 \\ h\cdot P_3A_1 & h\cdot P_3A_2 & 0 & -h\cdot P_3 & h\cdot P_3M \\ M^TP_1 & 0 & 0 & h\cdot M^TP_3 & -\varepsilon\cdot I \end{bmatrix}<0, \quad (10)$$

where $\tilde{\tilde{\Pi}}_{11}=A_1^TP_1+P_1A_1+P_2-Q_1-Q_1^T+\varepsilon\cdot N_1^TN_1$.

Now we provide a procedure to design a suitable H_∞ state feedback control.

Step 1: For the system (1) (resp. (7)), find the H_∞ control from Corollary 1 (resp. Corollary 2).

Step 2: Based on the above H_∞ control, we can use the less conservative criteria in Theorem 1 (resp. Theorem 2) to find the more useful result.

Step 3: If the obtained results in Step 1 and Step 2 are not satisfied the requirment for system performance. Then the genetic algorithm will be used for Theorem 1 (resp. Theorem 2) to find the control gain *K*, such that the minimization of γ can be achieved for every *K*; see for example [7].

3. Numerical examples

Example 1. Consider the system (1) with the parameters [10]:

$$A_1=\begin{bmatrix}0&0\\0&1\end{bmatrix},\ A_2=\begin{bmatrix}-1&-1\\0&-0.9\end{bmatrix},\ B_1=\begin{bmatrix}1\\1\end{bmatrix},$$
$$B_2=\begin{bmatrix}0\\1\end{bmatrix},\ C=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ D=\begin{bmatrix}0\\0.2\end{bmatrix}.$$

By the design procedure of H_∞ control with Theorem 1 and Corollary 1, we show this comparison in Table 1.

The control gains *K* and the disturbance attenuations (H_∞-norm bounds) γ for the results of this paper are smaller than the results in [10]. Larger state feedback gain *K* will cause the saturation in the amplifier applications. Smaller H_∞-norm bound γ will show the better effect on disturbance attenuation.

Example 2.

Consider the system (7) with the parameters [4]:

$$A_1=\begin{bmatrix}0&0\\0&1\end{bmatrix},\ A_2=\begin{bmatrix}-1&-1\\0&-0.9\end{bmatrix},\ B_1=\begin{bmatrix}0\\1\end{bmatrix},$$
$$B_2=\begin{bmatrix}1\\1\end{bmatrix},\ C=\begin{bmatrix}0&1\end{bmatrix},\ D=0,\ M=\begin{bmatrix}0.2&0\\0&0.2\end{bmatrix},$$
$$N_1=N_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ N_3=N_4=0.$$

By using the Corollary 2, we show the comparison in Table 2.

Example 3.

Consider the system (7a) with (A1), $u(t)=w(t)=B_1(t)=B_2(t)=0$, and the following parameters [4, 6]:

$$A_1=\begin{bmatrix}-2&0\\0&-0.9\end{bmatrix},\ A_2=\begin{bmatrix}-1&0\\-1&-1\end{bmatrix},$$
$$M=\begin{bmatrix}0.2&0\\0&0.2\end{bmatrix},\ N_1=N_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

The upper bounds of the time delay for the stability in [4] and [6] are $h=0.4437$ and $h=1.77$, respectively. By the Corollary 3 of this paper, the obtained upper bound for the time delay is $h=2.397$.

4. Conclusion

In this paper, the problem for the robust H_∞ control of time-delay systems is considered. LMI optimization approach has been developed to construct the H_∞ state feedback control. Some numerical examples have been given to demonstrate the potentials of our results.

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Table 1. Comparing the results of this paper with [10].

	$h = 0.95$		$h = 0.8$	
Results of [10]	$\gamma = 0.3856$	$K = [0 \quad 513780]$	$\gamma = 0.2882$	$K = [0 \quad 755020]$
Results of this paper	$\gamma = 0.2$	$K = [0 \quad 7674]$	$\gamma = 0.2$	$K = [0 \quad 7701]$

Table 2. Comparing the results of this paper with [4].

	$h = 0.3$	$h = 0.2$
Results of [4]	$\gamma = 1.95$, controller gain is not provided	$\gamma = 0.66$, controller gain is not provided
Results of this paper	$\gamma = 0.345$, $K = [-0.1263 \quad 3.7046]$	$\gamma = 0.2151$, $K = [-0.149 \quad 5.3425]$